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C. Atkinson and R. V. Craster

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The application of invariant integrals in diffusive elastic solids

By C. Atkinson[1]† and R. V. Craster[2]

[1] *Schlumberger Cambridge Research, High Cross, Madingley Road, Cambridge CB3 0EL, U.K.*

[2] *Department of Mathematics, Imperial College of Science, Technology and Medicine, London SW7 2BZ, U.K.*

Contents

Three techniques for deducing near crack tip singular fields from far field stress and pore pressure information are developed for the diffusive elastic theories of Biot: (*a*) methods based on a 'pseudo' energy–momentum tensor in the Laplace transformed domain; as a generalization of the energy–momentum tensor of Eshelby; (*b*) methods based on a reciprocal theorem in the Laplace transform domain; (*c*) methods based on a reciprocal theorem in real time.

All of the methods relate near crack tip singular fields to far field information.

In the most difficult cases, method (*a*) gives coefficients of singular stress fields and singular pore pressure gradients combined rather than separately. Nevertheless, this method is used to show that, remarkably, the complicated shear crack tip results derived by Craster & Atkinson can be checked in special circumstances.

Methods (*b*) and (*c*) require appropriate dual functions. Versions of these dual functions are determined. Combinations of all three methods can, of course, be used in conjunction with numerical methods. All three methods are illustrated first by using the diffusion equation and then by using the full poroelastic equations.

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† Permanent address: Department of Mathematics, Imperial College of Science, Technology and Medicine, London SW7 2BZ, U.K.

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0. Nomenclature

α	Biot's coefficient of effective stress, i.e. the ratio of fluid volume to the volume change of solid allowing the fluid to drain, where $0 < \alpha \leqslant 1$
B	Skempton's pore pressure coefficient (Skempton 1954), i.e. the ratio of induced pore pressure to the variation of mean normal compression under undrained conditions
c	generalized consolidation coefficient
δ_{ij}	Kronecker delta
e	dilatation
ϵ_{ij}	components of the strain tensor
κ	permeability coefficient
G	shear modulus
$\mathrm{d}\zeta/\mathrm{d}p\vert_\epsilon = 1/Q$	measure of the change in fluid content generated in a unit reference volume during a change of pressure with the strains kept constant
m	mass of fluid per unit volume
ν, ν_u	drained and undrained Poisson ratios, where $\nu \leqslant \nu_u \leqslant 0.5$
p	pore pressure, i.e. the increase in fluid pressure from a reference pressure p_0
Φ, Ψ	potential functions
q_i	mass flux vector
ρ_0	reference density
s, ξ	the Laplace and Fourier transform variables respectively
σ_{ij}, t_{ij}	stress tensor and elastic stress tensor respectively, i.e. $\sigma_{ij} = t_{ij} - \alpha p\, \delta_{ij}$
u_i	displacement vector
ζ	variation of fluid content per unit reference volume, i.e. mass of fluid per unit volume/initial density ρ_0

The following relations are used in the text:

$$\alpha = 3(\nu_u - \nu)/B(1-2\nu)(1+\nu_u), \tag{1}$$

$$Q = 2GB^2(1-2\nu)(1+\nu_u)^2/9(\nu_u-\nu)(1-2\nu_u), \tag{2}$$

$$\alpha Q = 2GB(1+\nu_u)/3(1-2\nu_u), \tag{3}$$

$$\eta = (1-\nu)/(1-2\nu), \tag{4}$$

$$\eta_u = (1-\nu_u)/(1-2\nu_u), \tag{5}$$

$$2G_u = 2GB(1+\nu_u)/3(1-\nu_u) = \alpha Q/\eta_u, \tag{6}$$

$$c = 2\kappa B^2 G(1-\nu)(1+\nu_u)^2/9(1-\nu_u)(\nu_u-\nu), \tag{7}$$

$$\bar{\eta} = (1-\nu)/2(\nu_u-\nu) = Gc/\kappa(1-\nu_u)(2G_u)^2. \tag{8}$$

For $\nu \leqslant \nu_u \leqslant 0.5$ then $\bar{\eta} \geqslant 1$.

$$\bar{\eta}_u = (1-\nu_u)/2(\nu_u-\nu), \tag{9}$$

$$N_0 = \tfrac{1}{2}(1-2\bar{\eta}). \tag{10}$$

Note the saturated, incompressible limit can be recovered by taking $\nu_u \to \frac{1}{2}$, $B \to 1$ with the results that $G_u \to G$, $\bar{\eta} \to \eta$, $\alpha \to 1$, $c \to 2G\eta\kappa$ and $Q \to \infty$.

1. Introduction

In previous papers by Atkinson & Craster (1991; see Appendix 5) and Craster & Atkinson (1992) (hereafter referred to as [AC] and [CA] respectively) solutions for quasi-static tensile and shear fracture in linear isotropic porous elastic and thermoelastic solids were obtained in the Laplace and Fourier transformed domain. The specific problem addressed was a semi-infinite crack opening under an impulsively applied load. The crack tip behaviour for an idealized loading was considered, with more complicated loadings following by superposition. As discussed in [CA] we are considering fracture in undamaged materials; the material ahead of the crack tip is continuous. The pore pressure conditions ahead of the fracture are set by symmetry. The crack faces may be either permeable or impermeable; the pores can be blocked by clay or the filter cake created in the hydraulic fracturing process.

The methods of [AC] and [CA] allow a fairly complete treatment of crack problems in infinite bodies (including the finite length crack problem which is approached in [AC] by singular perturbation theory). We now consider methods which can be used to analyse problems with finite boundaries and which can be used with numerical methods.

Two distinct methods are used to develop ancillary tools for extracting near crack tip stress and pore pressure fields from far field (or other non-singular) numerical data.

The first of these is based on a 'pseudo' energy–momentum tensor, which is a generalization of the elastic energy–momentum tensor Eshelby (1951, 1970). This is valid in the Laplace transform domain and has been developed by Atkinson & Smelser (1982) (for thermoviscoelasticity) and by Atkinson (1991) (for poroelasticity). In §3*a*(v) this method is applied in a non-trivial way to the problem of a shear crack in a strip.

The second method is based on using an appropriate reciprocal theorem and constructing suitable (auxiliary) dual functions. This method is known to the authors originally from the work of Barone & Robinson (1972) for elasticity, and has been applied by Stern and co-workers in a series of papers for various elastic problems (Stern *et al.* 1976). (See Atkinson (1983) for a review and Atkinson & Bastero (1991) for a recent application to anisotropic elastic bimaterials.) Whereas the methods based on the energy–momentum tensor are restricted to crack-like singularities, the reciprocal theorem method can be extended to wedge and notch singularities (see Atkinson (1984) for a review). For the poroelastic equations we give reciprocal theorems in the Laplace transform and real time domains (§2*b*(i) and §4). The key step is, however, the construction of appropriate dual functions and to determine them so as to make additional calculations in the far field as simple as possible. It should be stressed that for practical situations the method is to be used in conjunction with a numerical method; only in special circumstances would results be obtainable without numerical calculations. There are numerical boundary element (Cheng & Detournay 1988; Detournay & Cheng 1991) methods in use in the Laplace transform domain. Our Laplace transform methods will be of use in conjunction with those methods as well as the real time results.

The linear theories of isotropic thermoelasticity and poroelasticity were introduced and discussed in Biot (1941, 1955); in particular it is shown that in the quasi-static limit the two theories are mathematically equivalent. The theories introduce a direct coupling between the diffusing pore fluid (temperature) and the stress in the solid elastic skeleton (material).

The equations of thermoelasticity are usually uncoupled due to the small coupling parameter (Boley & Wiener 1960). This is not generally the case for poroelastic materials and the fully coupled equations need to be solved. Hence we deal with the fully coupled equations using the notation for the poroelastic equations as introduced in Rice & Cleary (1976).

The stress σ_{ij} is given by

$$\sigma_{ij} = 2G\epsilon_{ij} + 2G\nu(1-2\nu)^{-1}\delta_{ij}\epsilon_{kk} - \alpha p\delta_{ij} = t_{ij} - \alpha p \delta_{ij} \tag{11}$$

and the pore pressure p satisfies the linear relation,

$$p = Q\zeta - \alpha Q\epsilon_{kk}. \tag{12}$$

The governing equations, where we assume that there are no body forces or fluid sources in the body, are as follows:

(*a*) the equilibrium equation,

$$\sigma_{ij,j} = 0; \tag{13}$$

(*b*) Darcy's law, which relates the mass flux to the gradient of the pore pressure, where it is assumed that density fluctuations away from the reference density ρ_0 are small (analogous to the Fourier law of heat conduction for thermoelastic media),

$$q_i = -\rho_0 \kappa p_{,i}; \tag{14}$$

(*c*) the mass continuity equation (analogous to the entropy balance equation for thermoelasticity),

$$\partial m/\partial t = -q_{i,i}. \tag{15}$$

Here m is the mass of fluid per unit volume $= \zeta\rho_0$ and ρ_0 is a reference density. These can be combined and written as

$$\partial p/\partial t - \kappa Q \nabla^2 p = -\alpha Q\, \partial e/\partial t \tag{16}$$

and the Navier equation from (13) as

$$G\nabla^2 u_i + G(1-2\nu)^{-1} e_{,i} - \alpha p_{,i} = 0, \tag{17}$$

or alternatively in terms of the variation of fluid content ζ,

$$G\nabla^2 u_i + G(1-2\nu_u)^{-1} e_{,i} - \alpha Q \zeta_{,i} = 0, \tag{18}$$

$$c\nabla^2 \zeta = \partial \zeta/\partial t. \tag{19}$$

As noted in [CA] these equations are superficially uncoupled, the boundary conditions are usually given in terms of stresses, displacements and the pore pressure. The variation of fluid content is not a usual boundary condition, therefore the equations are still coupled through the boundary conditions. The equations are characterized by five independent constants: G, ν as in an elastic material, ν_u, α to characterize the interaction between solid and fluid constituents, and κ which characterizes the permeability of the material and the viscosity of the fluid.

2. Dual functions

In the work we present for dual functions, functions which can be combined in some way with the known solution or form of the solution in the reciprocal theorem in such a way as to deduce some required information, we will consider only the symmetric case, which corresponds to the tension or pressure driven fracture. A similar approach will work in the antisymmetric shearing case. Such an analysis would follow almost exactly that here, except that results from [CA] would be required. The basic idea is to generate eigensolutions of the governing equations, these are then combined with the 'real' solution in the reciprocal theorem. The singular behaviour of the 'real' solution in the neighbourhood of the crack tip is known from the eigenvalue analysis of [AC]; the eigensolutions are chosen so that the coefficients of this singular behaviour can then be extracted and related to far field integrals.

(a) *The diffusion equation*

As the following analysis is complex, it is valuable to illustrate the philosophy and practical application of what is to follow with a simple example using the diffusion equation,

$$\nabla^2 p = \partial p/\partial t. \tag{20}$$

Now assuming that the pressure is zero for $t<0$ we Laplace transform equation (20) with respect to time to get

$$\nabla^2 \bar{p}(x,y,s) = s\bar{p}(x,y,s). \tag{21}$$

If $\bar{p}$ and $\bar{p}'$ are two independent pressure fields satisfying (21) the following reciprocal theorem can be deduced

$$\int_S (\bar{p}'\,\bar{p}_{,j} - \bar{p}\bar{p}'_{,j})\, n_j \,\mathrm{d}S = 0 \tag{22}$$

for an arbitrary volume V with surface S. The notation $_{,j}$ denotes partial differentiation with respect to x_j.

The philosophy is to find eigensolutions (the primed field) which are singular, i.e. $\bar{p}' \sim O(r^{-\frac{1}{2}})$ in the neighbourhood of the crack tip and then use these together with the far field solution of a 'real' problem (perhaps obtained numerically) to deduce the near field behaviour of the 'real' problem.

(i) *Eigensolution*

We now solve (21) with the following boundary conditions on $y=0$

$$\bar{p}' = 0 \quad \text{for} \quad x<0 \quad \text{and} \quad \partial\bar{p}'/\partial y = 0 \quad \text{for} \quad x>0. \tag{23}$$

If we now Fourier transform with respect to x then (21) becomes

$$(\mathrm{d}^2/\mathrm{d}y^2 - \Gamma^2)\,\bar{\bar{p}}'(\xi,y,s) = 0 \tag{24}$$

with $\Gamma^2 = \xi^2 + s$. For solutions which decay as $y\to\infty$

$$\bar{\bar{p}}'(\xi,y,s) = A(\xi,s)\,\mathrm{e}^{-\Gamma y} \tag{25}$$

and Γ is taken to be the root with positive real part. To proceed, we introduce the following half-range Fourier transforms:

$$P_+ = \int_0^\infty \bar{p}'(x,0,s)\,\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x, \quad Q_- = \int_{-\infty}^0 \frac{\partial \bar{p}'(x,0,s)}{\partial y}\,\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x. \tag{26}$$

The subscripts $+$, $-$ are used to denote functions which are analytic in the upper and lower complex ξ planes respectively. From the boundary conditions (23) and (25) it is clear that

$$P_+ = A \quad \text{and} \quad Q_- = -\Gamma A. \tag{27}$$

Hence the following functional equation can be deduced:

$$Q_- = -\Gamma P_+. \tag{28}$$

To proceed further Γ is factorized into a product $\Gamma_+\Gamma_-$ where $\Gamma_\pm = (\xi \pm \mathrm{i}s^{\frac{1}{2}})^{\frac{1}{2}}$, i.e. they have branch cuts from $\mp \mathrm{i}s^{\frac{1}{2}}$ to $\mp \mathrm{i}\infty$. The functional equation (28) can now be split into $+$ and $-$ functions to give

$$Q_-/\Gamma_- = -\Gamma_+ P_+ = \Sigma(\xi), \tag{29}$$

where $\Sigma(\xi)$ is by analytic continuation an analytic function everywhere in the complex ξ plane. For our eigensolutions we require that as $r \to 0$ then $p' \sim O(r^{-\frac{1}{2}})$, which in transform space (see Appendix 1) means that $P_+ \sim O(\xi_+^{-\frac{1}{2}})$ as $|\xi| \to \infty$. Therefore by a simple application of Liouville's theorem $\Sigma(\xi)$ can be deduced to be a constant Σ. For convenience we shall take

$$\Sigma = -a(\pi \mathrm{i})_+^{\frac{1}{2}}/s \tag{30}$$

with a constant. The constant contains a $1/s$ term as the eigenfunctions correspond to an impulsive situation, i.e. all the field variables are zero for $t < 0$ after which the forcing (in the 'real' problem) is applied. Hence

$$\bar{p}'(x, y, s) = \frac{-\Sigma}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\Gamma y}}{\Gamma_+} \mathrm{e}^{\mathrm{i}\xi x} \, \mathrm{d}\xi, \tag{31}$$

which from the transform results in Appendix 1 gives our eigensolution $\bar{p}'$ as

$$\bar{p}'(x, y, s) = (a/sr^{\frac{1}{2}}) \cos(\tfrac{1}{2}\theta) \exp(-rs^{\frac{1}{2}}). \tag{32}$$

If the reciprocal theorem (22) is to be used then expression (32) is the required dual function, however, we can also find the eigensolutions explicitly using 29.3.83 of Abramowitz & Stegun (1970) (hereafter referred to as [A]) as

$$p'(x, y, t) = (a/r^{\frac{1}{2}}) \cos(\tfrac{1}{2}\theta) \operatorname{erfc}(r/2t^{\frac{1}{2}}) \tag{33}$$

and

$$\frac{\partial p'(x, y, t)}{\partial y} = \frac{-a}{2r^{\frac{3}{2}}} \operatorname{erfc}\left(\frac{r}{2t^{\frac{1}{2}}}\right) \sin(\tfrac{3}{2}\theta) - \frac{a}{(\pi r t)^{\frac{1}{2}}} \sin\theta \cos(\tfrac{1}{2}\theta)\, \mathrm{e}^{-r^2/4t}. \tag{34}$$

The analysis above is identical in spirit to that which will be outlined for the poroelastic case. However, for the diffusion equation we can solve this problem easily for a wedge or notch.

(ii) *Wedge problems*

We take the Laplace transformed diffusion equation (21) with

$$\bar{p}' = 0 \quad \text{on} \quad \theta = \pm\beta, \quad \partial \bar{p}'/\partial y = 0 \quad \text{on} \quad \theta = 0. \tag{35}$$

Since (21) is separable in cylindrical coordinates, using separation of variables and the symmetry of the problem we find

$$\bar{p}' = \sum_{n=0}^{\infty} (A_n(s) K_{\mu_n}(rs^{\frac{1}{2}}) + B_n(s) I_{\mu_n}(rs^{\frac{1}{2}})) \cos \mu_n \theta, \tag{36}$$

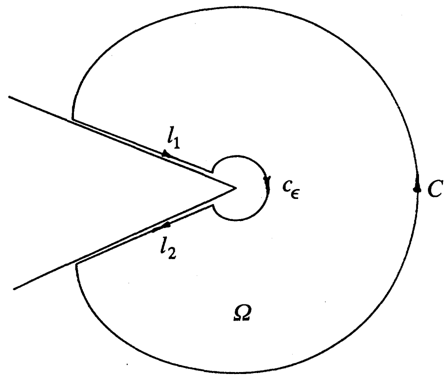

Figure 1. The contour for notch problems.

where $\mu_n = (2n+1)\pi/2\beta$ and the $K_n(z)$ and $I_n(z)$ are the modified Bessel's functions [A]. The eigensolutions which decay as $r \to \infty$ are given by

$$\bar{p}' = A_i(s) K_{\mu_i}(rs^{\frac{1}{2}}) \cos \mu_i \theta. \tag{37}$$

The eigensolution which is least singular at the origin is given by $i = 0$. If $\beta = \pi$ we recover the solution of §2*a*(i), where we note from 10.2.17 of [A] that $K_{\frac{1}{2}}(z) = (\pi/2z)^{\frac{1}{2}} \mathrm{e}^{-z}$, and that the asymptotic behaviour of the modified Bessel's function $K_\nu(z)$ as $z \to 0$ is given by $K_\nu(z) \sim \frac{1}{2}\gamma(\nu)(\frac{1}{2}z)^{-\nu}$; $\gamma(\nu)$ is the gamma function defined in Appendix 1.

(iii) *Near field evaluation*

For the real solution where the crack or wedge is pressure free, it is clear from an eigenvalue analysis of the diffusion equation that as $r \to 0$, $\nabla^2 \bar{p} \sim 0$. Hence to leading order in the neighbourhood of the crack or wedge tip

$$\bar{p} \sim \sum_{n=0}^{\infty} C_n(s)\, r^{\mu_n} \cos \mu_n \theta. \tag{38}$$

It is the intensity factors $C_n(s)$ that we wish to evaluate.

Consider the wedge in figure 1, and let C be the contour of radius R along which some pressure loading is applied, c_ϵ be the contour of radius ϵ enclosing the crack tip and l_1, l_2 be the contours along the wedge faces. Then substituting $\bar{p}', \bar{p}$ from (37), (38) in the reciprocal theorem (22)

$$\int_{c_\epsilon} \left(\bar{p}' \frac{\partial \bar{p}}{\partial r} - \bar{p} \frac{\partial \bar{p}'}{\partial r} \right) r \,\mathrm{d}\theta = -\int_C \left(\bar{p}' \frac{\partial \bar{p}}{\partial r} - \bar{p} \frac{\partial \bar{p}'}{\partial r} \right) r \,\mathrm{d}\theta. \tag{39}$$

Note that the integrals are zero along the wedge faces as $\bar{p}' = \bar{p} = 0$ there. Taking the limit as $\epsilon \to 0$ in the first integral, and using the orthogonality of the eigensolutions and asymptotic solutions gives

$$C_n(s) = \frac{(\frac{1}{2}s^{\frac{1}{2}})^{\mu_n}}{\beta \mu_n \gamma(\mu_n)} \int_{-\beta}^{\beta} r \left(K_{\mu_n}(rs^{\frac{1}{2}}) \frac{\partial \bar{p}}{\partial r} - \bar{p} \frac{\mathrm{d}K_{\mu_n}(rs^{\frac{1}{2}})}{\mathrm{d}r} \right) \Bigg|_{r=R} \cos(\mu_n \theta) \,\mathrm{d}\theta. \tag{40}$$

The recurrence relations for the derivatives of modified Bessel's functions are given in Appendix 1.

It is anticipated that the 'real' solution $\bar{p}$ is evaluated using some numerical scheme of finite elements or boundary elements whose accuracy near the crack tip is not clear. However, the numerical data should be more accurate in the far field, and it is this data which is then substituted into (40) to deduce the correct near field behaviour. The numerical evaluation of the stress intensity factors is a difficult and delicate process and it is envisaged that the extension of this simple example will be useful though it is not pursued here.

This example illustrates the general procedure; note that all the above has been performed in Laplace transform space. We now proceed to consider the more complex problem of poroelastic fracture using a similar approach.

(b) *Poroelastic fracture*

For poroelastic tensile fracture we consider the case of a stress free crack in a porous elastic material. The material is in equilibrium at $t = 0$ when some stress and/or pressure field is applied in the far field. The crack is on $y = 0$, $x < 0$ and the applied loadings are assumed symmetric; for more general cases with shear loadings the eigenfunctions from the shear situation [CA] are needed.

If the material is continuous ahead of the crack tip we have two possible cases. Firstly the crack faces ($y = 0$, $x < 0$) may be permeable $\bar{p} = 0$ or they may be impermeable $\partial\bar{p}/\partial y = 0$. From the symmetry of the tensile problem, the condition on the x-axis ahead of the crack will be that $\partial\bar{p}/\partial y = 0$ for $x > 0$. The situation with mixed pore pressure conditions is more complex, therefore it is the case we treat in detail here; the method will also work for the unmixed boundary conditions, i.e. impermeable crack faces $\partial\bar{p}/\partial y = 0$ for all x.

(i) *Reciprocal theorem*

In Laplace transform space a reciprocal theorem can be deduced for linear poroelasticity (Cleary 1977). If we assume that the primed and unprimed variables are two independent poroelastic states, there are no body forces or sources and that there is no pre-existing pressure or stress field, then the reciprocal theorem is

$$\int_S \left((\bar{\sigma}_{ij}\,\bar{u}'_i - \bar{\sigma}'_{ij}\,\bar{u}_i) + \frac{\kappa}{s}(\bar{p}\bar{p}'_{,j} - \bar{p}'\bar{p}_{,j}) \right) n_j \,\mathrm{d}S = 0 \tag{41}$$

for an arbitrary volume V with surface S. The overbar denotes Laplace transformed quantities.

(ii) *Eigensolutions*

To deduce the eigensolutions we follow the procedure introduced in §2*a*(i). In the mixed case the 'real' solutions will, in general, near the crack tip have a singular stress field characterized by a stress intensity factor $K_1(t)$ and a singular pore pressure gradient characterized by $K_2(t)$. It is our object to identify these in terms of integrals which can then be evaluated.

As in [AC] the governing equations and the variables of interest can be written in terms of the Biot potentials (Biot 1956). The details are given in [AC] and the explicit form of the displacements, stresses and pressure field are given in Appendix 3. The equations for the potentials can now be written as

$$\nabla^2 \Psi = 0, \quad c\nabla^4 \Phi = \partial \nabla^2 \Phi / \partial t \tag{42}$$

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and hence when these equations are Laplace transformed

$$\nabla^2 \bar{\Psi} = 0, \quad \nabla^4 \bar{\Phi} = (s/c)\nabla^2 \bar{\Phi}. \tag{43}$$

We introduce the following scaling:

$$X = x(s/c)^{\frac{1}{2}} \quad \text{and} \quad Y = y(s/c)^{\frac{1}{2}}, \tag{44}$$

$$\bar{\sigma}_{ij}(x,y,s) = \bar{T}_{ij}(X,Y,s)\,(s/c)^{\frac{1}{2}}, \quad \bar{\zeta}(x,y,s) = \bar{V}(X,Y,s)\,(s/c)^{\frac{1}{2}}, \tag{45}$$

$$\bar{u}_i(x,y,s) = \bar{U}_i(X,Y,s) \quad \text{and} \quad \bar{p}(x,y,s) = \bar{P}(X,Y,s)\,(s/c)^{\frac{1}{2}}, \tag{46}$$

$$\bar{\Phi}'(X,Y,s) = \bar{\Phi}(x,y,s)\,(s/c)^{\frac{1}{2}} \quad \text{and} \quad \bar{\Psi}'(X,Y,s) = \bar{\Psi}(x,y,s)\,(s/c)^{\frac{1}{2}}. \tag{47}$$

This scales s out of the Laplace transformed equations. The resulting equations from (18), (19) are:

$$G\nabla^2_{XY}\bar{U}_i + G(1-2\nu_u)^{-1}\bar{E}_{,i} - \alpha Q\bar{V}_{,i} = 0, \tag{48}$$

$$\nabla^2_{XY}\bar{V} = \bar{V}, \tag{49}$$

and

$$\nabla^2_{XY}\bar{\Psi}' = 0, \quad \nabla^4_{XY}\bar{\Phi}' = \nabla^2_{XY}\bar{\Phi}', \tag{50}$$

where $\nabla^2_{XY} = \partial^2/\partial X^2 + \partial^2/\partial Y^2$ and $_{,i}$ denotes differentiation with respect to X_i. Now, taking the Fourier transform with respect to X, i.e.

$$\bar{\Phi}''(\xi,Y,s) = \left(\frac{s}{c}\right)^{\frac{1}{2}} \int_0^{\infty}\int_{-\infty}^{+\infty} \bar{\Phi}(x,y,t)\,\mathrm{e}^{\mathrm{i}\xi X - st}\,\mathrm{d}X\,\mathrm{d}t \tag{51}$$

gives from (50) the equations,

$$(\mathrm{d}^2/\mathrm{d}Y^2 - \Gamma^2)(\mathrm{d}^2/\mathrm{d}Y^2 - \xi^2)\,\bar{\Phi}''(\xi,Y,s) = 0, \tag{52}$$

$$(\mathrm{d}^2/\mathrm{d}Y^2 - \xi^2)\,\bar{\Psi}''(\xi,Y,s) = 0, \tag{53}$$

where $\Gamma^2 = \xi^2 + 1$. Considering the upper half-plane and assuming that the stresses tend to zero as $Y \to \infty$ implies that:

$$\bar{\Phi}''(\xi,Y,s) = A_1(\xi,s)\,\mathrm{e}^{-|\xi|Y} + A_2(\xi,s)\,\mathrm{e}^{-\Gamma Y}, \tag{54}$$

$$\bar{\Psi}''(\xi,Y,s) = B_1(\xi,s)\,\mathrm{e}^{-|\xi|Y}, \tag{55}$$

$|\xi| = \xi_+^{\frac{1}{2}}\,\xi_-^{\frac{1}{2}}$ with both square roots real and positive for ξ real and positive and $\xi_\pm^{\frac{1}{2}}$ having branch cuts in the complex ξ plane from 0 to $\mp\mathrm{i}\infty$ respectively.

In [AC] the following problem was solved.

A semi-infinite crack on $y = 0$, $x < 0$ has the following boundary conditions on $y = 0$;

$$\sigma_{xy} = 0\;\forall x, \tag{56}$$

$$\sigma_{yy} = -\tau_0\,\mathrm{e}^{x/a}H(t) \quad \text{when} \quad x < 0, \tag{57}$$

where $H(t)$ is the Heaviside step function. This corresponds to an impulsively loaded crack under an internal stress. The form of the loading is somewhat artificial, but it can be used to generate more general loadings by superposition. We also assume

$$p = 0 \quad \text{when} \quad x < 0, \tag{58}$$

i.e. a crack with permeable crack faces. From the symmetry we require that

$$u_2 = 0 \quad \text{and} \quad \partial p/\partial y = 0 \quad \text{when} \quad x > 0. \tag{59}$$

A modification to the solution of this problem gives us the eigensolutions required. We take half-range transforms as in the example of [AC] and then modify the solution of the resulting functional equations to pick out the eigensolutions. Since we follow the analysis of [AC], taking $\tau_0 = 0$ there. Only the outline of the solution is given here.

We define T_+ and P_+ to be the half-range Fourier transforms of $\bar{T}_{YY}$ and $\bar{P}$, on $Y = 0, X > 0$, respectively

$$T_+ = \int_0^{+\infty} \bar{T}_{YY}(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X, \quad P_+ = \int_0^{+\infty} \bar{P}(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X \tag{60}$$

and define U_-, R_- to be the half-range transforms of $\bar{U}_2, \partial\bar{P}/\partial Y$ on $Y = 0, X < 0$ respectively,

$$U_- = \int_{-\infty}^0 \bar{U}_2(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X, \quad R_- = \int_{-\infty}^0 \frac{\partial\bar{P}}{\partial Y}(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X. \tag{61}$$

Here the subscripts $+, -$ denote functions which are regular in the upper and lower complex ξ planes respectively.

Using the potential representations in Appendix 3 it is possible to combine the half-range transforms in such a way as to form a functional equation in terms of $+, -$ functions. Following [AC] let

$$M_+ = T_+ + 3P_+/2B(1+\nu_u), \quad L_- = R_- + 2G_u\,\xi^2\,U_-, \tag{62}$$

$$N = \xi^2 - |\xi|\Gamma + \bar{\eta}, \tag{63}$$

the potentials give

$$A_1|\xi| + A_2\Gamma = 0, \tag{64}$$

$$P_+ = 2Q\alpha\xi^2 B_1 + (c/\kappa)A_2, \tag{65}$$

$$R_- = -2Q\alpha\xi^2|\xi|B_1 - (c/\kappa)\Gamma A_2, \tag{66}$$

$$T_+ = 2G(-\xi^2 B_1/(1-2\nu_u) + (G_u/G)A_2(\xi^2 - |\xi|\Gamma)), \tag{67}$$

$$U_- = 2\eta_u|\xi|B_1. \tag{68}$$

The following equation can be deduced

$$M_+ = (-2G_u\kappa/c)L_- N/\Gamma. \tag{69}$$

To separate this into $+, -$ functions, $N(\xi)$ and Γ need to be split into products of functions analytic in the upper and lower complex ξ planes. Following our simple example earlier, it is clear that Γ can be factorized into a product $\Gamma_+\Gamma_-$ where $\Gamma_\pm = (\xi \pm \mathrm{i})^{\frac{1}{2}}$, i.e. they have branch cuts from $\mp\mathrm{i}$ to $\mp\mathrm{i}\infty$. The product split for $N(\xi)$ is performed in Appendix 1 of [AC] and relevant results are reproduced in Appendix 2 here. From (69)

$$\Gamma_+ M_+/N_+ = -2G_u\kappa L_- N_-/c\Gamma_- = \Sigma(\xi). \tag{70}$$

Hence it can be shown that

$$A_2 = \Sigma/2G_u N_-\Gamma_+. \tag{71}$$

By using the definitions (62), (63), $\Sigma(\xi)$ can be extended by analytic continuation as an analytic function in the whole complex ξ plane.

For the following analysis we can deduce two eigensolutions, one of which is too singular for a physical problem in both the stress and pore pressure field. From the reciprocal theorem we can then deduce a relationship between K_1, K_2 (the mode 1 stress intensity factor and the coefficient of the (singular) pore pressure gradient; see

Appendix 4) and a far field integral by using one eigensolution. By using the other eigensolution, which is just singular enough in the pore pressure to act as a dual function for the pore pressure itself, then gives K_2 related to the far field. To generate the first-order eigensolutions we set $\Sigma = a + b\xi$. (Note that we could pick out higher-order eigensolutions by letting $\Sigma = c\xi^2 + d\xi^3 \ldots$ and choosing the appropriate edge conditions for the pore pressure.) Defining

$$k(\xi) = 1/N_-(\xi)\,\Gamma_+(\xi)\xi_+^{\frac{1}{2}} \tag{72}$$

This function is split into a sum of + and − functions in Appendix 2 of [AC]; some useful results are quoted in Appendix 2 here. Hence we can deduce directly from [AC] or from (71), (68), (65) that

$$\frac{P_+}{\xi_+^{\frac{1}{2}}} - \frac{ck_+(\xi)\,\Sigma(\xi)}{2G_u\,\kappa} = 2G_u\,\xi_-^{\frac{1}{2}}\,U_- + \frac{ck_-(\xi)\,\Sigma(\xi)}{2G_u\,\kappa} = J_1(\xi). \tag{73}$$

Here by analytic continuation $J_1(\xi)$ is regular in the whole complex ξ plane. Taking the limit as $|\xi| \to \infty$ and using the asymptotic properties of N_+, N_-, k_+, k_- as given in Appendix 2, together with the appropriate edge conditions, gives J_1. From [AC] or (71), (67), (66) the following secondary equation can also be deduced:

$$\begin{aligned} T_+\,\xi_+^{\frac{1}{2}} - N_+\,\xi_+^{\frac{1}{2}}\,\Sigma(\xi)/\Gamma_+ + \bar{\eta}\Sigma(\xi)\,\xi(k_+ - c_0) \\ = 3R_-/2B(1+\nu_u)\,\xi_-^{\frac{1}{2}} - \bar{\eta}\Sigma(\xi)\,(\xi(k_- + c_0) - \Gamma_-/N_-\,\xi_-^{\frac{1}{2}}) = J_2(\xi), \end{aligned} \tag{74}$$

where $J_2(\xi)$ is by analytic continuation an analytic function in the whole complex ξ plane. It is determined in a similar manner to J_1.

(iii) *First eigensolution*

The first eigensolution is chosen to be singular in both the stress and pore pressure gradient so that the eigensolutions have edge conditions $\bar{\sigma}'_{yy} \sim O(r^{-\frac{3}{2}})$, $\bar{u}'_2 \sim O(r^{-\frac{1}{2}})$, $\partial\bar{p}'/\partial y \sim O(r^{-\frac{3}{2}})$ and $\bar{p}' \sim O(r^{-\frac{1}{2}})$ as $r \to 0$. In transform space (see Appendix 1) this implies that as $|\xi| \to \infty$, $T'_+ \sim O(\xi_+^{\frac{1}{2}})$, $U' \sim O(\xi_-^{-\frac{1}{2}})$, $P'_+ \sim O(\xi_+^{-\frac{1}{2}})$ and $R'_- \sim O(\xi_-^{\frac{1}{2}})$. To get this eigensolution we take $a = 0$.

Near field. As $|\xi| \to \infty$ we have from the edge conditions $M_+ \sim T'_+$ using the asymptotic behaviour of $\Gamma_+ = (\xi + \mathrm{i})^{\frac{1}{2}} \to \xi_+^{\frac{1}{2}}$ and $N_+ \to 1$ (Appendix 2), we can deduce immediately from (70) that

$$T'_+ \sim b\xi_+^{\frac{1}{2}}. \tag{75}$$

A simple application of Liouville's theorem in (73) gives,

$$J_1(\xi) = (-bc/2G_u\,\kappa)\,(c_0\,\xi + (N_+(0)/\bar{\eta} - d)) \tag{76}$$

and therefore

$$P'_+ \sim (-cb/2G_u\,\kappa\xi_+^{\frac{1}{2}})\,(\mathrm{i}N_+(0)/2\bar{\eta} + \varpi), \tag{77}$$

$$U'_- \sim \frac{-b(1-\nu)}{G}\,\xi_-^{-\frac{1}{2}} - \frac{cb}{(2G_u)^2\,\kappa}\left(\varpi + \tfrac{1}{2}\mathrm{i}\left(\frac{N_+(0)}{\bar{\eta}} + \frac{1}{N_0} + \frac{I}{N_0}\right)\right)\xi_-^{-\frac{3}{2}} + \ldots. \tag{78}$$

Using the above result for U_- we can deduce from (70)

$$R'_- \sim (bc/2G_u\,\kappa)\,(\varpi + \mathrm{i}N_+(0)/2\bar{\eta})\,\xi_-^{\frac{1}{2}}. \tag{79}$$

Hence using Liouville's theorem in (74) $J_2(\xi) = -\bar{\eta}b\xi(d - N_+(0)/\bar{\eta})$. (The relevant results from [AC] are reproduced in Appendix 2.)

The asymptotic results invert, using the scalings and the results in Appendix 1 to give, on the x-axis,

$$\bar{p}' \sim \left(\frac{-cb}{2G_u\kappa}\right)\left(\frac{\mathrm{i}N_+(0)}{2\bar{\eta}}+\varpi\right)\left(\frac{s}{c}\right)^{\frac{1}{4}}\frac{1}{(\pi \mathrm{i}x)_+^{\frac{1}{2}}} \quad \text{for} \quad x>0, \tag{80}$$

$$\bar{u}_2' \sim -(b(1-\nu)\,\mathrm{i}_+^{\frac{1}{2}}/G)\,\pi^{-\frac{1}{2}}(c/s)^{\frac{1}{4}}(-x)^{-\frac{1}{2}} \quad \text{for} \quad x<0, \tag{81}$$

$$\bar{\sigma}_{yy}' \sim -b(\mathrm{i}_+^{\frac{1}{2}}/2\pi^{\frac{1}{2}})\,(c/s)^{\frac{1}{4}}x^{-\frac{3}{2}} \quad \text{for} \quad x>0, \tag{82}$$

$$\frac{\partial \bar{p}'}{\partial y} \sim \frac{-bc}{2G_u\kappa}\left(\frac{\mathrm{i}N_+(0)}{2\bar{\eta}}+\varpi\right)\left(\frac{s}{c}\right)^{\frac{1}{4}}\frac{1}{2(\pi \mathrm{i})_+^{\frac{1}{2}}}(-x)^{-\frac{3}{2}} \quad \text{for} \quad x<0. \tag{83}$$

Taking the singular pore pressure field for a crack (Appendix 4),

$$\bar{p}' \sim K_2'(s)\,(2\pi r)^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta, \tag{84}$$

we deduce that

$$\bar{p}' \sim \left(\frac{-cb}{2G_u\kappa}\right)\left(\frac{\mathrm{i}N_+(0)}{2\bar{\eta}}+\varpi\right)\left(\frac{s}{c}\right)^{\frac{1}{4}}\frac{1}{(\pi \mathrm{i}r)_+^{\frac{1}{2}}}\cos\tfrac{1}{2}\theta. \tag{85}$$

For the stresses and displacements take $\beta=\pi$, $\lambda=\frac{1}{2}$ in Appendix 4 and $K_1(s)$ there to be $K_1'(s)$ (the prime being used to denote the first eigensolution) which is deduced to be

$$K_1'(s) = b\mathrm{i}_+^{\frac{1}{2}}\,2^{-\frac{1}{2}}\,(c/s)^{\frac{1}{4}}. \tag{86}$$

The stresses and displacements are then given by (272)–(276).

Far field. We also require the behaviour of the far fields of the eigensolutions for use in the reciprocal theorem. These are deduced by matching the solutions from (73), (74) with an outer solution.

To deduce the asymptotic behaviour in the far field we note that from the Laplace transform of (16)

$$\kappa Q\nabla^2\bar{p}' = s(\alpha Q\bar{e}'+\bar{p}'). \tag{87}$$

In the far field

$$\bar{p}' \sim -\alpha Q\bar{e}'. \tag{88}$$

Thus to leading order, $\bar{\zeta}'$, the Laplace transformed variation of fluid content, is effectively zero out in the far field and hence

$$G\nabla^2\bar{u}_i'+G(1-2\nu_u)^{-1}\,\bar{e}_{,i}' \sim 0, \tag{89}$$

i.e. to leading order we have the usual elastic solution (with undrained coefficients). The displacements must remain finite as $r\rightarrow\infty$; hence we expect that $\bar{u}_i \sim O(r^{-\frac{1}{2}})$ at most. The eigensolutions in the far field are therefore given by matching with the inverted transform fields as

$$\bar{p}' \sim 2(CB(1+\nu_u)/3\surd(2\pi r^3))\cos\tfrac{3}{2}\theta \tag{90}$$

and the elastic eigensolution (with undrained coefficients) is deduced from Appendix 4, taking $\beta=\pi$, $\lambda=\frac{1}{2}$ and $K_1(s)$ there is given by GC.

C can be deduced as

$$C = \tfrac{1}{2}b\bar{\eta}(2\mathrm{i})_+^{\frac{1}{2}}\,(N_+(0)/\bar{\eta}-d)\,(s/c)^{-\frac{1}{4}}. \tag{91}$$

Application of the reciprocal theorem. We consider the contour of figure 2, and apply

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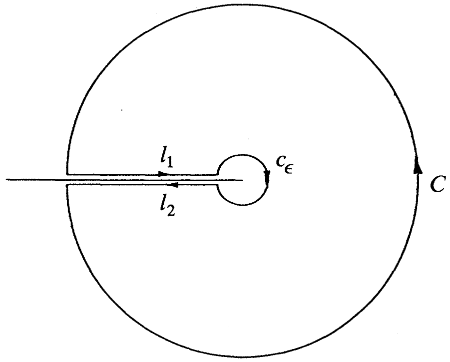

Figure 2. The contour for crack problems.

the reciprocal theorem with the primed field as the above eigensolution, and the unprimed field as the 'real' solution. As the crack is assumed to be permeable and stress free, the integrals along the crack faces are zero. The crack tip integral can be evaluated using the asymptotic results for the crack tip fields deduced in Appendix 4, the near field results, and the far field integral with the far field solution. Hence

$$\frac{2(1-\nu)}{G}K_1(s)\,K_1'+\frac{\kappa}{2s}K_2(s)\,K_2' = -\int_{-\pi}^{\pi}((\bar{\sigma}_{ir}\,\bar{u}_i'-\bar{\sigma}_{ir}'\,\bar{u}_i)+\frac{\kappa}{s}(\bar{p}\bar{p}'_{,r}-\bar{p}'\bar{p}_{,r}))\,r|_{r=R}\,\mathrm{d}\theta. \quad (92)$$

This gives a relation between K_1 and K_2 for the 'real' solution in terms of an integral which can be evaluated.

(iv) *Second eigensolution*

This eigensolution is chosen to be one which is singular in the pore pressure, but has the usual stress singularity. Hence we take the edge conditions as $r \to 0$ to be that $\bar{\sigma}''_{yy} \sim O(r^{-\frac{1}{2}})$, $\bar{u}''_2 \sim O(r^{\frac{1}{2}})$, $\bar{p}'' \sim O(r^{-\frac{1}{2}})$ and $\partial\bar{p}''/\partial y \sim O(r^{-\frac{3}{2}})$; so as $|\xi| \to \infty$, T''_+, $P''_+ \sim O(\xi_+^{-\frac{1}{2}})$, $U''_- \sim O(\xi_-^{-\frac{3}{2}})$, $R''_- \sim O(\xi_-^{\frac{1}{2}})$. The double prime has been added here to prevent any confusion with the previous eigensolution. We take $\Sigma(\xi)$ above to be constant (i.e. $b = 0$) and hence from the secondary Wiener–Hopf equations (73), (76) we can deduce the asymptotic behaviour of the variables of interest.

Near field. In the limit as $|\xi| \to \infty$ we apply Liouville's theorem to (73) to deduce

$$J_1(\xi) = -cc_0\,a/2G_u\,\kappa \quad (93)$$

and hence
$$P''_+ \sim (ca/2G_u\,\kappa)\,(N_+(0)/\bar{\eta}-d)\,\xi_+^{-\frac{1}{2}}. \quad (94)$$

From the definition of M_+ and from (70) we can deduce

$$T''_+ \sim a(1-\bar{\eta}\,(N_+(0)/\bar{\eta}-d))\,\xi_+^{-\frac{1}{2}} \quad (95)$$

and hence from (76) that $J_2(\xi) = 0$. The near crack tip behaviour of the pore pressure is given from Appendix 4 as

$$\bar{p}'' \sim \frac{ac}{2G_u\,\kappa}\left(\frac{N_+(0)}{\bar{\eta}}-d\right)\frac{(s/c)^{\frac{1}{4}}}{(\mathrm{i}\pi)_+^{\frac{1}{2}}}r^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta = \frac{K''_2}{(2\pi r)^{\frac{1}{2}}}\cos\tfrac{1}{2}\theta. \quad (96)$$

(As the near crack tip stress field for this eigensolution is $O(r^{-\frac{1}{2}})$ its contribution to the near field integral in the reciprocal theorem will be zero.)

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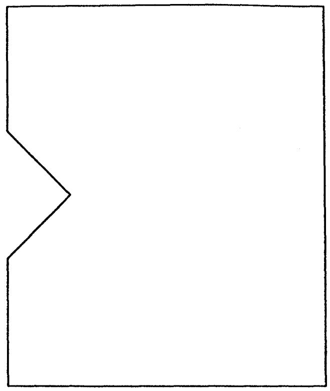

Figure 3. A finite region.

Far field. The far field solutions are deduced in a similar manner to those of the first eigensolution. As the displacements must remain finite as $r \to \infty$. Hence $\bar{u}'' \sim O(r^{-\frac{1}{2}})$. The eigensolutions in the far field are therefore given by matching with the inverted transform fields as

$$\bar{p}'' \sim 2\,(CB(1+\nu_u)/3\sqrt{(2\pi r^3)})\cos(\tfrac{3}{2}\theta), \tag{97}$$

$$\bar{u}''_\theta = \tfrac{1}{4}(C/\sqrt{(2\pi r)})\,(-(5-8\nu_u)\sin(\tfrac{3}{2}\theta)+3\sin\tfrac{1}{2}\theta), \tag{98}$$

$$\bar{u}''_r \sim \tfrac{1}{4}(C/\sqrt{(2\pi r)})\,((7-8\nu_u)\cos(\tfrac{3}{2}\theta)-3\cos\tfrac{1}{2}\theta), \tag{99}$$

$$\bar{\sigma}''_{\theta\theta} \sim -\tfrac{1}{4}(GC/\sqrt{(2\pi r^3)})\,(\cos(\tfrac{3}{2}\theta)+3\cos\tfrac{1}{2}\theta), \tag{100}$$

$$\bar{\sigma}''_{r\theta} \sim -\tfrac{1}{4}(3GC/\sqrt{(2\pi r^3)})\,(\sin(\tfrac{3}{2}\theta)+\sin\tfrac{1}{2}\theta), \tag{101}$$

$$\bar{\sigma}''_{rr} \sim -\tfrac{1}{4}(GC/\sqrt{(2\pi r^3)})\,(7\cos\tfrac{3}{2}\theta-3\cos\tfrac{1}{2}\theta). \tag{102}$$

Here C is given by

$$C = \tfrac{1}{2}c_0\,a(2\mathrm{i})^{\frac{1}{2}}_{+}\bar{\eta}\,(s/c)^{-\frac{1}{4}}, \tag{103}$$

i.e. the elastic eigensolution (with undrained coefficients) in Appendix 4 with $\beta = \pi$, $\lambda = \frac{1}{2}$ and taking $K_1(s)$ there to be equal to GC.

Application of the reciprocal theorem. We take the surface S to be the contour of figure 2. The integrals along the crack faces are zero as the pore pressure in both the 'real' and eigensolution are zero; similarly the crack is stress free for both solutions. Therefore

$$\frac{\kappa}{2s}K_2(s)\,K''_2 = -\int_{-\pi}^{\pi}((\bar{\sigma}_{ir}\,\bar{u}'_i-\bar{\sigma}'_{ir}\,\bar{u}_i)+\frac{\kappa}{s}(\bar{p}\bar{p}'_{,r}-\bar{p}'\bar{p}_{,r}))\,r|_{r=R}\,\mathrm{d}\theta. \tag{104}$$

This gives $K_2(s)$ as an integral which can be evaluated.

(c) *Numerical methods*

Although we have worked through the theory for a particular, rather specialized case, we can apply the same method for other cases. For instance, the notch of figure 3. In this case we need the primed field which is in transform space and the transforms need to be inverted. This inversion cannot be done analytically and must be done numerically. This can be achieved in an accurate and rapid manner by utilizing the fast Fourier transform (Brigham 1973). Let

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\xi)\,e^{-i\xi x}\,d\xi. \tag{105}$$

If $F(\xi)$ tends to zero as $|\xi|\rightarrow\infty$ (which it must if the integral is well defined), we can approximate $f(x)$ by

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} \frac{\pi}{l} F\left(\frac{n\pi}{l}\right) \exp\,(-in\pi x/l). \tag{106}$$

Replacing x by $m\Delta x = 2lm/N$ and truncating the sum gives

$$f(m\Delta x) = \frac{1}{2\pi}\sum_{n=-N/2}^{N/2-1} F\left(\frac{n\pi}{l}\right)\frac{\pi}{l} \exp\,(-2\pi i n m/N). \tag{107}$$

Changing the summation range gives

$$f(\Delta x(m'-\tfrac{1}{2}N)) = \frac{1}{N\Delta x}\sum_{n'=0}^{N-1} (-1)^{n'+m'} F\left(\frac{\pi}{l}(n'-\tfrac{1}{2}N)\right) \exp\,(-2\pi i n' m'/N), \tag{108}$$

which is exactly the formulation for the inverse fast Fourier transform routines. Here $2l$ is the length of the interval (*ca.* 50) and N is the number of intervals (*ca.* 1000). A useful check is to invert $e^{-\Gamma y}/\Gamma_+$, which is inverted exactly in Appendix 1, (203). The inversion was done using double precision complex arithmetic and NAG routine C06EAF. For small values of y the exact solution is $O(r^{-\frac{1}{2}})$ as $r\rightarrow 0$; as a consequence the numerical result contains some oscillatory 'noise' in this region and this can be filtered out. Note when using complex arithmetic with functions containing branch cuts, it is important to check which branches the machine takes. The numerical analysis is in the Laplace transform domain and it is important to remember the scaling introduced earlier.

(*d*) *Separation of variables*

An alternative approach for generating auxiliary solutions is to note that the Laplace transformed equations (43) are separable in cylindrical polars. From the symmetry of the tensile problems we drop the $\sin\lambda\theta$ terms and terms in θ, $\ln r$. To get the following series solutions

$$\bar{\Psi}(r,\theta,s) = \sum_{n=0}^{\infty} \cos\chi_n\theta(A_n r^{\chi_n} + \bar{A}_n r^{-\chi_n}), \tag{109}$$

$$\bar{\Phi}(r,\theta,s) = \sum_{n=0}^{\infty} -(c/s)\cos\omega_n\theta(C_n r^{\omega_n} + \bar{C}_n r^{-\omega_n}) + (B_n K_{\mu_n}(r(s/c)^{\frac{1}{2}}) + \bar{B}_n I_{\mu_n}(r(s/c)^{\frac{1}{2}}))\cos\mu_n\theta, \tag{110}$$

where the μ_n, ω_n, χ_n are eigenvalues determined from the boundary conditions and $K_n(z), I_n(z)$ are the modified Bessel's functions [A]. Using the equations in Appendix 3 this gives the following two cases.

The impermeable crack where

$$\overline{\partial p/\partial y} = 0, \quad \text{on} \quad \theta = 0, \pi, \tag{111}$$

by taking $\omega_n = n$, $\mu_n = n$ and $\chi_n = \frac{1}{2}(2n+1)$. This satisfies all the boundary conditions except for $\bar{\sigma}_{xy}, \bar{\sigma}_{yy} = 0$ on $\theta = \pi$.

The permeable crack where

$$\bar{p} = 0 \quad \text{on} \quad \theta = \pi, \quad \partial\bar{p}/\partial y = 0 \quad \text{on} \quad \theta = 0 \tag{112}$$

by taking $\omega_n = \frac{1}{2}(2n+1), \mu_n = \frac{1}{2}(2n+1)$, $\chi_n = \frac{1}{2}(2n+1)$. This choice satisfies all the boundary conditions except for $\bar{\sigma}_{xy} = 0$ on $\theta = \pi$.

These solutions satisfy the poroelastic equations and hence they can be used as auxiliary functions in the reciprocal theorem. However, they do not satisfy the boundary conditions exactly, so that there is a price to be paid for the simpler structure of these eigensolutions. When they are substituted into the reciprocal theorem, the integral along the crack faces is no longer zero, and this integration now has to be performed.

(i) *Wedge problems*

If we are prepared to perform the integration along the crack faces we can also consider the following auxiliary function.

$$\bar{\Psi} = 0, \tag{113}$$

$$\bar{\Phi} = A\cos\left(\frac{\pi\theta}{2\beta}\right)\left(K_{\pi/2\beta}\left(r\left(\frac{s}{c}\right)^{\frac{1}{2}}\right) - \frac{\gamma(\pi/2\beta)r^{-\pi/2\beta}}{2(\frac{1}{2}(s/c)^{\frac{1}{2}})^{\pi/2\beta}}\right) - B\frac{c}{s}r^{\lambda+1}\cos(\lambda+1)\theta, \tag{114}$$

where λ is a zero of $\sin 2\lambda\beta + \lambda\sin(2\beta)$, which gives

$$\bar{p}' = (s/\kappa)AK_{\pi/2\beta}(r(s/c)^{\frac{1}{2}})\cos(\pi\theta/2\beta), \tag{115}$$

i.e. the solution to the diffusion equation we considered earlier. Let

$$F(r) = (K_{\pi/2\beta}(r(s/c)^{\frac{1}{2}}) - \gamma(\pi/2\beta)r^{-\pi/2\beta}/2(\tfrac{1}{2}(s/c)^{\frac{1}{2}})^{\pi/2\beta}), \tag{116}$$

then

$$\bar{u}_r' = \frac{G_u}{G}\left(A\cos\left(\frac{\pi\theta}{2\beta}\right)\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{Bc}{s}(\lambda+1)r^{\lambda}\cos(\lambda+1)\theta\right), \tag{117}$$

$$\bar{u}_\theta' = \frac{G_u}{G}\left(-\frac{\pi AF(r)}{2\beta r}\sin\left(\frac{\pi\theta}{2\beta}\right) + \frac{Bc}{s}r^{\lambda}(\lambda+1)\sin(\lambda+1)\theta\right), \tag{118}$$

$$\bar{\sigma}_{rr}' = -2G_u\left(A\cos\left(\frac{\pi\theta}{2\beta}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\left(\frac{\pi}{2\beta}\right)^2\right)F(r) + \frac{Bc}{s}\lambda(\lambda+1)r^{\lambda-1}\cos(\lambda+1)\theta\right), \tag{119}$$

$$\bar{\sigma}_{\theta\theta}' = -2G_u\left(A\cos\left(\frac{\pi\theta}{2\beta}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}\right)F(r) - \frac{Bc}{s}(\lambda+1)^2\cos(\lambda+1)\theta r^{\lambda-1}\right), \tag{120}$$

$$\bar{\sigma}_{r\theta}' = 2G_u\left(\frac{-A\pi}{2\beta r}\sin\left(\frac{\pi\theta}{2\beta}\right)\frac{\mathrm{d}F(r)}{\mathrm{d}r} + \frac{Bc}{s}(\lambda+1)^2 r^{\lambda-1}\sin(\lambda+1)\theta\right). \tag{121}$$

This is clearly not an ideal eigensolution, as on the wedge faces $\bar{\sigma}_{r\theta}'$ and $\bar{\sigma}_{\theta\theta}'$ are both non-zero. (For a crack ($\beta = \pi$) then only $\bar{\sigma}_{r\theta}'$ is non-zero.)

(ii) *The reciprocal theorem for the crack*

If $\beta = \pi$ then the situation is somewhat simplified; in particular we know that

$$K_{\frac{1}{2}}(z) = (\pi/2z)^{\frac{1}{2}}\mathrm{e}^{-z} \tag{122}$$

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and that $\lambda = -\frac{1}{2}$ in the above. As $r \to 0$ we note that

$$F(r) = K_{\frac{1}{2}}(r(s/c)^{\frac{1}{2}}) - (\pi/2r(s/c)^{\frac{1}{2}})^{\frac{1}{2}} \sim -(\tfrac{1}{2}\pi r(s/c)^{\frac{1}{2}})^{\frac{1}{2}}. \tag{123}$$

Therefore using the reciprocal theorem

$$-\int_{-\pi}^{\pi}\left((\bar{\sigma}_{ir}\,\bar{u}'_i - \bar{\sigma}'_{ir}\,\bar{u}_i) + \frac{\kappa}{s}(\bar{p}\bar{p}'_{,r} - \bar{p}'\bar{p}_{,r})\right) r\,\mathrm{d}\theta$$

$$= \frac{\bar{K}_1(s)(2\pi)^{\frac{1}{2}} G_u}{8G}\left(A(\tfrac{1}{2}\pi)^{\frac{1}{2}}\left(\frac{s}{c}\right)^{\frac{1}{4}} + \frac{Bc}{s}\right) + \tfrac{1}{2}\bar{K}_2(s)\,A\,\pi\left(\frac{c}{s}\right)^{\frac{1}{4}} - 2\int_0^{\infty} \bar{\sigma}'_{r\theta}\,\bar{u}_{\theta|\theta=\pi}\,\mathrm{d}x \tag{124}$$

as $\bar{\sigma}_{r\theta}$ and $\bar{u}_\theta$ are both antisymmetric. Setting $A = 0$, we can deduce $\bar{K}_1(s)$ and then taking arbitrary values for A, B find $\bar{K}_2(s)$.

3. Invariant integral for poroelasticity

If all the field variables are assumed zero for $t < 0$ then we Laplace transform the governing equations (13), (16) to get

$$\bar{\sigma}_{ij,j} = 0, \tag{125}$$

$$\kappa Q \nabla^2 \bar{p} = s(\alpha Q \bar{e} + \bar{p}), \tag{126}$$

as in Atkinson (1991). Consider the lagrangian L

$$L = -\tfrac{1}{2}\bar{t}_{ij}\,\bar{e}_{ij} + \alpha\overline{p u}_{i,i} + (\kappa/2s)\,\bar{p}_{,i}\,\bar{p}_{,i} + \bar{p}^2/2Q. \tag{127}$$

We recall t_{ij} is defined to be the elastic part of the stress tensor. The Euler–Lagrange equations are equivalent to (125), (126), i.e.

$$\partial/\partial x_j\,(\partial L/\partial \bar{u}_{i,j}) - \partial L/\partial \bar{u}_i = 0 \to \bar{\sigma}_{ij,j} = 0, \tag{128}$$

$$\partial/\partial x_j\,(\partial L/\partial \bar{p}_{,j}) - \partial L/\partial \bar{p} = 0 \to \kappa Q \bar{p}_{,jj} = s(\alpha Q \bar{u}_{i,i} + \bar{p}). \tag{129}$$

In Atkinson (1991) the pseudo energy–momentum tensor is defined as

$$\bar{P}_{lj} = \partial L/\partial \bar{u}_{j,l}\,\bar{u}_{i,l} + \partial L/\partial \bar{p}_{,j}\,p_{,l} - L\delta_{lj} \tag{130}$$

giving from (127) that

$$\bar{P}_{lj} = -\bar{\sigma}_{ij}\,\bar{u}_{i,l} + (\kappa/s)\,\bar{p}_{,j}\,\bar{p}_{,l} - L\delta_{lj}. \tag{131}$$

As L does not depend explicitly upon x_l, $\bar{P}_{lj,j} = 0$ and hence the integrals

$$F_l = \int_S \bar{P}_{lj}\,n_j\,\mathrm{d}S \quad \text{for} \quad l = 1,\,2,\,3 \tag{132}$$

are zero when S is a closed surface enclosing no singularities.

These equations are analogous to the usual formulation for elasticity; however, the equations are all in the transform domain so have no direct physical significance.

(*a*) *The application of the invariant integral*

In the Laplace transform domain the tensor $\bar{P}_{lj}$ can be used with the poroelastic equations in much the same way as in thermoviscoelasticity (Atkinson & Smelser 1982). However, as described there, the simplest invariant F_1 leads in the most difficult cases to a result for the near field which involves a combination (as the sum of the squares) of the stress intensity factor and the coefficient of the singular pore pressure (temperature) gradient term at the crack tip. Such formulae could be used together with the dual function approach to get explicit results. There are also non-trivial cases where the $\bar{P}_{lj}$ method itself will lead to the explicit determination of near field behaviour. We shall illustrate the method firstly with the diffusion equation, and then with the transient shear problem as considered in [CA].

(i) *The diffusion equation*

In a similar manner to that above we can deduce for the diffusion equation that $\bar{P}_{lj}$ is given by

$$\bar{P}_{lj} = s^{-1}\bar{p}_{,j}\bar{p}_{,l} - \delta_{lj}\left(\tfrac{1}{2}s^{-1}\bar{p}_{,i}\bar{p}_{,i} + \tfrac{1}{2}\bar{p}^2\right). \tag{133}$$

We now consider the antisymmetric (loaded) analogue of §2*a*(i): the problem of a semi-infinite cut where on $y = 0$

$$\bar{p} = 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad \partial\bar{p}/\partial y = -q_0\,\mathrm{e}^{x/a} \quad \text{for} \quad x < 0. \tag{134}$$

Assuming that the pressure decays as $y \to \infty$ then if we consider just the upper half-plane

$$\bar{p}(\xi, y, s) = A(\xi, s)\,\mathrm{e}^{-\Gamma y}. \tag{135}$$

To proceed we introduce the following half-range Fourier transforms

$$Q_+ = \int_0^\infty \frac{\partial\bar{p}(x,0,s)}{\partial y}\,\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x, \quad P_- = \int_{-\infty}^0 \bar{p}(x,0,s)\,\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x. \tag{136}$$

The subscripts $+$, $-$ are used to denote functions which are analytic in the upper and lower complex ξ planes respectively. From the boundary conditions it is clear that

$$P_- = A, \quad Q_+ - q_0 a/s(1+\mathrm{i}\xi a) = -\Gamma A. \tag{137}$$

Hence noting the simple pole at i/a the following functional equation can be deduced

$$\frac{Q_+}{\Gamma_+} - \frac{q_0 a}{s(1+\mathrm{i}\xi a)}\left(\frac{1}{\Gamma_+} - \frac{1}{\Gamma_+(\mathrm{i}/a)}\right) = -\Gamma_- P_- + \frac{q_0 a}{s(1+\mathrm{i}\xi a)\,\Gamma_+(\mathrm{i}/a)} = 0 \tag{138}$$

giving as $|\xi| \to \infty$, i.e. at the tip of the cut (as $x \to 0-$, $y = 0$)

$$P_- \sim (q_0/\mathrm{i}s\Gamma_+(\mathrm{i}/a))\,\xi_-^{-\frac{3}{2}}. \tag{139}$$

From the diffusion equation we know that in the neighbourhood of the tip (as $r \to 0$)

$$\bar{p} \sim \bar{K}_3(s)\,(r/2\pi)^{\frac{1}{2}} \sin\tfrac{1}{2}\theta. \tag{140}$$

Hence we can deduce $\bar{K}_3(s)$ as

$$\bar{K}_3(s) = q_0 \mathrm{i}^{\frac{1}{2}}_{+} 2\surd 2/s\Gamma_+(\mathrm{i}/a) \tag{141}$$

and as $a \to \infty$, i.e. a uniform load, this remains finite and

$$\bar{K}_3(s) \to q_0\, 2\surd 2/s^{\frac{5}{4}}. \tag{142}$$

(ii) *The finite width strip*

We now consider the problem of a finite width strip (figure 4). As well as giving simple direct results, we can use this solution to check the above result, using $\bar{P}_{lj}$, and taking the limit as the width of the strip tends to infinity. We assume antisymmetric

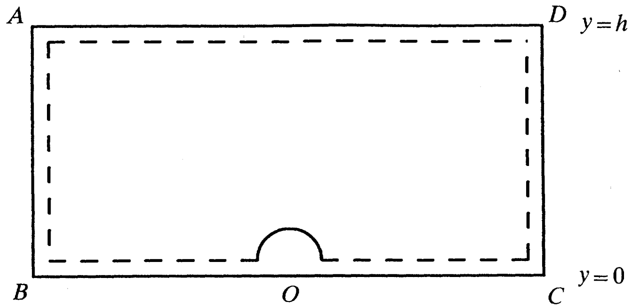

Figure 4. The contour for use with the invariant integral.

loadings about $y=0$ and consider only the upper half of the strip. Take as a boundary condition that

$$\bar{p}=p_0 f(s) \quad \text{on} \quad y=h, \tag{143}$$

$$\bar{p}=0 \quad \text{on} \quad y=0, x>0, \partial\bar{p}/\partial y=0 \quad \text{on} \quad y=0, x<0, \tag{144}$$

then as $x \to \pm\infty$, the solutions there are given by

$$\mathrm{d}^2\bar{p}/\mathrm{d}y^2+s\bar{p}=0. \tag{145}$$

The resulting solution which satisfies the boundary conditions as $x\to\infty$, i.e. on CD is

$$\bar{p}=p_0 f(s)\sinh(s^{\frac{1}{2}}y)/\sinh(s^{\frac{1}{2}}h) \tag{146}$$

and as $x\to-\infty$ on AB

$$\bar{p}=p_0 f(s)\cosh(s^{\frac{1}{2}}y)/\cosh(s^{\frac{1}{2}}h). \tag{147}$$

Now using the contour $ABOCD$ and the property that the integral

$$F_1=\int_S \bar{P}_{1j}\, n_j\, \mathrm{d}S=0 \tag{148}$$

since it does not enclose any singularities, we can deduce that

$$\bar{K}_3^2(s)=16p_0^2 f(s)^2 s^{\frac{1}{2}}/\sinh(2s^{\frac{1}{2}}h). \tag{149}$$

(iii) *Comparison with infinite problem*

For the semi-infinite crack in the strip with a uniform loading, the boundary conditions on $y=0$ are that

$$\bar{p}=0 \quad \text{on} \quad x>0 \quad \text{and} \quad \partial\bar{p}/\partial y=-q_0/s \quad \text{for} \quad x<0, \tag{150}$$

also $\bar{p}=0$ on $y=\pm h$.

We subtract off the solution of the strip without the cut (i.e. the solution as $x\to\infty$ (146)) from the problem considered in §3*a* (ii), the tip field will be unaltered. We find that

$$\partial\bar{p}/\partial y=-p_0 f(s)\, s^{\frac{1}{2}}\cosh(s^{\frac{1}{2}}y)/\sinh(s^{\frac{1}{2}}h) \tag{151}$$

as $x\to-\infty$. Hence, by comparison, with (150)

$$q_0/s=s^{\frac{1}{2}}p_0 f(s)/\sinh(s^{\frac{1}{2}}h). \tag{152}$$

Therefore taking the limit as $h\to\infty$, i.e. the infinite body, in (149) and substituting for $p_0 f(s)$ from (152) the result (142) is duplicated. It is also worthwhile noting here that a similar approach can be used for bimaterial strip problems.

(iv) *The poroelastic equations*

Consider the semi-infinite crack transient shear problem of [CA] and suppose that instead of the shear stress loading considered there, there is now a boundary condition on the pore pressure gradient; i.e. on $y = 0$

$$\sigma_{xy} = 0, \quad \partial p/\partial y = -q_0\,\mathrm{e}^{x/a} H(t) \quad \text{for} \quad x < 0 \tag{153}$$

and that ahead of the crack due to the anti-symmetry

$$p = 0, \quad u_1 = 0 \quad \text{for} \quad x > 0, \tag{154}$$

and finally that on $y = 0$, $\sigma_{yy} = 0$. The solution follows that of [CA] closely and is only sketched very briefly here. The shear potentials from Appendix 3 are used, and the same Laplace transform method and scalings as in §2*b*(ii). The half range transforms of $\bar{T}_{XY}$ and $\partial\bar{P}/\partial Y$ are τ_+ and R_+ respectively, on $Y = 0$, $X > 0$, i.e.

$$\tau_+ = \int_0^{+\infty} \bar{T}_{XY}(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X, \quad R_+ = \int_0^{+\infty} \frac{\partial \bar{P}(X,0,s)}{\partial Y}\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X \tag{155}$$

and U_-, P_- are the half-range transforms of $\bar{U}_1, \bar{P}$ on $Y = 0$, $X < 0$, i.e.

$$U_- = \int_{-\infty}^{0} \bar{U}_1(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X, \quad P_- = \int_{-\infty}^{0} \bar{P}(X, 0, s)\,\mathrm{e}^{\mathrm{i}\xi X}\,\mathrm{d}X. \tag{156}$$

The subscripts $+$, $-$ denote functions which are regular in the upper and lower complex ξ planes.

The boundary conditions can be written as

$$\bar{T}_{XY} = 0, \quad \partial\bar{P}/\partial Y = -q_0 a_1 c/s^2(1 + \mathrm{i}\xi a_1) + R_+, \tag{157}$$

$\bar{T}_{YY} = 0, \bar{U}_1 = U_-, \bar{P} = P_-$ all on $y = 0$. Using the formulae in Appendix 3 and the boundary conditions above gives a system of equations which relate A_1, A_2 and B_1 to the physical quantities on the X-axis. These are treated in [CA]. The main result is that the mode 2 stress intensity factor is given by

$$\bar{K}_{II}(s) = \frac{-3q_0 c\sqrt{2}}{\mathrm{i}_+^{\frac{1}{2}}\, s^2(\bar{d} + 1/\bar{N}_-(0))\, a_1\, 2B(1+\nu_u)\,\bar{\eta}} \left(\frac{1}{(\mathrm{i}/a_1)_+^{\frac{3}{2}}} + \frac{\bar{\eta}\bar{k}_+(\mathrm{i}/a_1)}{\bar{N}_+(\mathrm{i}/a_1)\,\Gamma_+(\mathrm{i}/a_1)} \right) \left(\frac{s}{c}\right)^{\frac{1}{4}}. \tag{158}$$

The interesting result is that with this purely pore pressure gradient loaded fracture, we can take the limit as $a \to \infty$, i.e. uniformly loaded along the entire crack. The stress intensity factor is finite

$$\bar{K}_{II}(s) = \frac{2^{\frac{1}{2}}\mathrm{i}c3q_0 \bar{K}_+(0)}{s^2(\bar{d} + 1/\bar{N}_-(0))\, 2B(1+\nu_u)\,\bar{N}_+(0)} \left(\frac{s}{c}\right)^{\frac{1}{4}}. \tag{159}$$

We can also deduce the pore pressure gradient coefficient $\bar{K}_3(s)$ in this limit as

$$\bar{K}_3(s) = \frac{-2\mathrm{i}q_0\sqrt{2}}{s\bar{N}_+(0)\,(\bar{d} + 1/\bar{N}_-(0))} \left(\left(\frac{1}{2\bar{N}_-(0)} + \bar{\chi} \right) \bar{\eta}\bar{K}_+(0) - \mathrm{i}\bar{\eta} \left(\bar{d} + \frac{1}{\bar{N}_-(0)} \right)^2 \right) \left(\frac{s}{c}\right)^{-\frac{1}{4}}. \tag{160}$$

(v) *A crack in a finite width strip*

We now consider the problem of a finite width strip with pressure conditions imposed on the sides $y = \pm h$. Again direct results can be obtained using the invariant integral (132) and we can check the above results using $\bar{P}_{lj}$ (131) after taking the limit as the width of the strip tends to infinity. Taking the boundary conditions to be

$$\bar{u}_1 = \bar{u}_2 = 0 \quad \text{and} \quad \bar{p} = \pm p_0 f(s) \quad \text{on} \quad y = \pm h, \tag{161}$$

$$\bar{p} = 0 \quad \text{on} \quad y = 0, x > 0, \quad \partial\bar{p}/\partial y = 0 \quad \text{on} \quad y = 0, x < 0, \tag{162}$$

$$\bar{u}_1 = 0 \quad \text{on} \quad y = 0, x > 0, \quad \bar{\sigma}_{xy} = 0 \quad \text{on} \quad y = 0, x < 0, \tag{163}$$

$$\bar{\sigma}_{yy} = 0 \quad \text{on} \quad y = 0, \tag{164}$$

then as $x \to \pm\infty$ the solutions there are given by

$$\kappa Q \,\mathrm{d}^2\bar{p}/\mathrm{d}y^2 - \alpha Q s\bar{e} - s\bar{p} = 0, \tag{165}$$

$$\bar{\sigma}_{i2,2} = 0, \tag{166}$$

which implies that $\bar{u}_1 = 0$ as $x \to \pm\infty$. The above equations can be solved using the boundary conditions (161)–(164) to give, as $x \to \infty$ along CD.

$$\bar{p} = p_0 f(s) \sinh((s/c)^{\frac{1}{2}} y)/\sinh((s/c)^{\frac{1}{2}} h), \tag{167}$$

$$\bar{u}_2 = \frac{\alpha p_0 f(s)(1-2\nu)}{2G(s/c)^{\frac{1}{2}}(1-\nu)\sinh((s/c)^{\frac{1}{2}} h)}(\cosh((s/c)^{\frac{1}{2}} y) - \cosh((s/c)^{\frac{1}{2}} h)), \tag{168}$$

and, as $x \to -\infty$ along AB,

$$\bar{p} = p_0 f(s) \cosh((s/c)^{\frac{1}{2}} y)/\cosh((s/c)^{\frac{1}{2}} h)), \tag{169}$$

$$\bar{u}_2 = \frac{\alpha p_0 f(s)(1-2\nu)}{2G(s/c)^{\frac{1}{2}}(1-\nu)\cosh((s/c)^{\frac{1}{2}} h)}(\sinh((s/c)^{\frac{1}{2}} y) - \sinh((s/c)^{\frac{1}{2}} h)). \tag{170}$$

Using the expression for $\bar{P}_{1j}$ (131), the asymptotic results as $r \to 0$ (140), and that part of F_1 has the same form as the elastic energy release rate (with drained coefficients) in the neighbourhood of the crack, integrating around the contour in figure 4 gives

$$\frac{\kappa}{s}\frac{\bar{K}_3^2(s)}{16} - \frac{\bar{K}_{II}^2(s)(1-\nu)}{4G} = \frac{p_0^2 \kappa f(s)^2}{c(s/c)^{\frac{1}{2}} \sinh(2(s/c)^{\frac{1}{2}} h)}. \tag{171}$$

To connect with the solution in (159) and (160) we first need to subtract off the solution of the strip without the crack. This is the field as $x \to \infty$, i.e. the only non-zero pressure and displacement are $\bar{p}$ and $\bar{u}_2$ as given in (167), (168) above. As a consequence the field at the crack tip should be the same as that of a strip with $\bar{p} = 0$ on its sides, but with $\partial\bar{p}/\partial y = -q_0/s$ on the crack $y = 0$, $x < 0$ where

$$q_0/s = (s/c)^{\frac{1}{2}} p_0 f(s)/\sinh((s/c)^{\frac{1}{2}} h). \tag{172}$$

It is now possible to let $h \to \infty$ in the full strip solution to give the result

$$\frac{\kappa}{s}\frac{\bar{K}_3^2(s)}{16} - \frac{\bar{K}_{II}^2(s)(1-\nu)}{4G} = \frac{q_0^2 \kappa}{2s^3(s/c)^{\frac{1}{4}}}, \tag{173}$$

which must correspond to the problem of §3*a*(iv). We can now check results (159), (160) numerically. The material constants, etc., drop out of the above equation leaving us to show that

$$\left(\frac{\bar{\eta}}{\bar{N}_+(0)\,(\bar{d}+1/\bar{N}_-(0))}\right)^2 \left(\mathrm{i}\bar{K}_+(0)\left(\bar{\chi}+\frac{1}{2\bar{N}_-(0)}\right)+\left(\bar{d}+\frac{1}{\bar{N}_-(0)}\right)^2\right)^2 - \bar{\eta}\left(\frac{\mathrm{i}\bar{K}_+(0)}{\bar{d}+1/\bar{N}_-(0)}\right)^2 = 1. \tag{174}$$

The integrals were evaluated by using NAG routines D01AJF, D01AKF (NAG 1991) and (174) was verified for several values of ν, ν_u, providing a useful independent check on the results in [CA].

As an aside we note that one of the current theories in earthquake mechanics as discussed in Stark & Stark (1991) is that large pore pressure gradients caused by rising pore fluid pressure provide a mechanism for brittle failure. The problem considered in §3*a*(v) is an impermeable stress free crack in a poroelastic strip. The boundary conditions on the edges of the strip ($y = \pm h$) are that the displacements are fixed, and that there is an impulsively applied pore pressure at the edges which is of opposite sign on $y = \pm h$ respectively. Physically this would correspond to a stress free fault in a layer of porous material between two rigid substrates. If the upper substrate was a reservoir of fluid and the lower substrate behaved as a sink, we would have the boundary conditions of §3*a*(v). Using the argument following (171) we can deduce that if the width of this layer then tends to infinity the Laplace transformed shear stress intensity factor is given by (159). In real time $K_{II}(t) \sim t^{\frac{3}{4}}$, if the pore pressure gradient is applied for sufficiently long the failure criterion will be exceeded. Hence, the scenario discussed above provides a mechanism for shear faulting, the rapid redistribution of fluid which is often observed following earthquakes is then a result of the fracture propagation. This could be modelled using the methods of [CA], this redistribution of fluid would then trigger the smaller aftershocks which often follow the main earthquake.

The invariant integral can also be used in a very similar manner to verify the results in [AC], consider a pore pressure loaded crack in a finite width strip and follow the method above. In [CA] the invariant integral is used together with the matching ideas of [AC] to verify the small time results obtained for the intensity factors in [CA].

4. Auxiliary functions and the real time reciprocal theorem

In previous sections we have considered a 'dual function' approach to enable near wedge (or crack) tip singular behaviour to be deduced in the Laplace transform domain from far field numerical information. Such solutions are directly useful when numerical or other approximate analysis has been completed in the Laplace transform domain. There are such analyses available (Booker 1973; Cheng & Detournay 1988), so the results of the previous sections should be useful. It is also worth stressing that the results of §2*b* have the property that only the far field information is necessary. The functions as introduced in §2*d* require both far field and boundary information to calculate the near tip singular fields, but as a consequence are much simpler to implement. In the representation we have been prepared to pay the price of near tip boundary evaluations in the interest of simpler dual functions, and to extend the range of problems from cracks to notches. Hence it is natural to enquire whether a real time formulation is possible, given that some relaxation of the constraints required of the dual functions (e.g. satisfaction of all boundary conditions) is allowed. There are also some numerical methods which are in real time (Zienkiewicz 1984; Dargush & Banerjee 1991)) so this analysis should be of value there.

Once again we will illustrate the method using the diffusion equation and then apply a similar procedure to the full poroelastic equations.

(a) *The diffusion equation*

Consider the region as shown in figure 1 with boundary C, area Ω and the thermal diffusion equation

$$\kappa\nabla^2 T = \partial T/\partial t, \tag{175}$$

where $\kappa = k/\rho c$, k is the thermal conductivity, ρ the density and c the specific heat of the solid and the following boundary conditions:

$$T = 0 \quad \text{on the wedge faces,} \tag{176}$$

$$aT + b\,\partial T/\partial n = f(r,\theta,t) \quad \text{on} \quad C. \tag{177}$$

Here the loading at infinity or remote loading on the outer (finite) boundary is for convenience and assumed to be symmetric about $\theta = 0$.

$$T = 0 \quad \text{at} \quad t = 0. \tag{178}$$

Multiplying by a function T^* we can deduce that

$$\int_0^\tau \int_\Omega \left(\kappa\nabla^2 T^* + \frac{\partial T^*}{\partial t}\right) T - \left(\kappa\nabla^2 T - \frac{\partial T}{\partial t}\right) T^* \,\mathrm{d}\Omega\,\mathrm{d}t$$

$$= \int_\Omega TT^*\,\mathrm{d}\Omega|_{t=0}^{t=\tau} + \int_0^\tau \int_C \left(T\frac{\partial T^*}{\partial n} - T^*\frac{\partial T}{\partial n}\right)\mathrm{d}C\,\mathrm{d}t. \tag{179}$$

In general to deduce the appropriate dual functions we would need to evaluate

$$\kappa\nabla^2 T^* + \partial T^*/\partial t = 0 \tag{180}$$

subject to

$$T^* = 0 \text{ on the wedge faces,} \tag{181}$$

$$T^* = 0 \quad \text{at} \quad t = 0. \tag{182}$$

If this has been done the result is

$$\int_\Omega TT^*\,\mathrm{d}\Omega|_{t=0}^{t=\tau} + \int_0^\tau \int_C \left(T\frac{\partial T^*}{\partial n} - T^*\frac{\partial T}{\partial n}\right)\mathrm{d}C\,\mathrm{d}t = 0. \tag{183}$$

As $T = 0$ at $t = 0$ we can consider a solution T^* which is time independent, i.e.

$$T^* = r^{-\pi/2\beta}\cos(\pi\theta/2\beta). \tag{184}$$

From §2*a* (iii) we have for the full solution as $r \to 0$ that

$$T \sim K_2(t) r^{\pi/2\beta}\cos(\pi\theta/2\beta). \tag{185}$$

Hence we can deduce

$$\pi\int_0^\tau K_2(t)\,\mathrm{d}t = \int_\Omega T r^{-\pi/2\beta}\cos(\pi\theta/2\beta)\,\mathrm{d}\Omega|^{t=\tau} + \int_0^\tau\int_C\left(T\frac{\partial T^*}{\partial n} - T^*\frac{\partial T}{\partial n}\right)\mathrm{d}C\,\mathrm{d}t. \tag{186}$$

(b) *The poroelastic equation*

Consider the same region as before, figure 1, and the governing equations in the form

$$\kappa Q\nabla^2 p = \alpha Q\,\partial e/\partial t + \partial p/\partial t, \tag{187}$$

$$\sigma_{ij,j} = 0. \tag{188}$$

The following boundary conditions are assumed

$$p=0,\quad \sigma_{r\theta}=0, \sigma_{\theta\theta}=0 \text{ on the wedge faces,} \tag{189}$$

$$ap+b\,\partial p/\partial n=f(r,\theta,t), \sigma_{rr}, \sigma_{r\theta} \text{ specified on } C, \tag{190}$$

and for convenience the remote loadings are assumed to be symmetric. Initially we assume zero remote pressure and stress fields

$$p=0,\quad \sigma_{ij}=0 \quad \text{at} \quad t=0 \tag{191}$$

and we use the assumed symmetry to set

$$\partial p/\partial y=0,\quad u_\theta=0,\quad \sigma_{r\theta}=0 \quad \text{on} \quad \theta=0. \tag{192}$$

Various 'pseudo' reciprocal theorems can be written down. Using functions p' and σ'_{ij} we can deduce that

$$\int_0^\tau\int_\Omega(\sigma'_{ij}\,u_{i,j}-\sigma_{ij}\,u'_{i,j})\,\mathrm{d}\Omega\,\mathrm{d}t=\int_0^\tau\int_\Omega\alpha(p'e-pe')\,\mathrm{d}\Omega\,\mathrm{d}t=\int_0^\tau\int_C(\sigma'_{ij}\,u_i-\sigma_{ij}\,u'_i)\,n_j\,\mathrm{d}C\,\mathrm{d}t \tag{193}$$

and

$$\int_0^\tau\int_\Omega\left(Q\kappa\nabla^2p'-\alpha Q\frac{\partial e'}{\partial t}+\frac{\partial p'}{\partial t}\right)p-\left(Q\kappa\nabla^2p-\alpha Q\frac{\partial e}{\partial t}-\frac{\partial p}{\partial t}\right)p'\,\mathrm{d}\Omega\,\mathrm{d}t$$
$$=\int_0^\tau\int_C\kappa Q\left(\frac{\partial p'}{\partial x_j}p-\frac{\partial p}{\partial x_j}p'\right)n_j\,\mathrm{d}C\,\mathrm{d}t+\int_\Omega pp'|_0^\tau\,\mathrm{d}\Omega+\int_0^\tau\int_\Omega\alpha Q\left(p'\frac{\partial e}{\partial t}-p\frac{\partial e'}{\partial t}\right)\mathrm{d}\Omega\,\mathrm{d}t. \tag{194}$$

There are both satisfied if the primed functions satisfy

$$\sigma'_{ij,j}=0,\quad \sigma'_{ij}=t'_{ij}-\alpha p', \tag{195}$$

$$Q\kappa\nabla^2p'-\alpha Q\,\partial e'/\partial t+\partial p'/\partial t=0. \tag{196}$$

As the material is in equilibrium at $t=0$ we take the dual fields to be time independent. We take the dual solution in (193) to be the purely elastic eigensolution $O(r^{-\lambda-1})$ from Appendix 4, with $p'=0$ everywhere. The result is that

$$-\int_\Omega\alpha pe'\,\mathrm{d}\Omega\,\mathrm{d}t=\int_C(\sigma'_{ij}\,u_i-\sigma_{ij}\,u'_i)\,n_j\,\mathrm{d}C. \tag{197}$$

The dual solution in (194) is chosen to be the field which is singular in the pressure

$$p'=r^{-\pi/2\beta}\cos(\pi\theta/2\beta). \tag{198}$$

The elastic eigensolution in (194) is not used as the time independence removes it.

$$\pi\int_0^\tau K_2(t)\,\mathrm{d}t=\int_\Omega(p+\alpha Qe)\,p'|_0^\tau\,\mathrm{d}\Omega+\int_0^\tau\int_C\kappa Q\left(p\frac{\partial p'}{\partial n}-p'\frac{\partial p}{\partial n}\right)\mathrm{d}C\,\mathrm{d}t. \tag{199}$$

5. Conclusion

Ancillary methods have been discussed for deducing the coefficients of singular near crack and notch tip stress fields and pore pressure gradients. Methods based on reciprocal theorems in Laplace transform and real time domains have been discussed.

For both of these methods auxiliary 'dual' functions are required, which enable the near tip singular behaviour to be determined, and, by satisfying appropriate boundary conditions, allow the integrals involved in the reciprocal theorem to be moved far from the crack or notch tip. For the case of notch problems in poroelasticity, where the pore pressure and elastic fields are fully coupled, such 'complete' dual functions are difficult to determine, and simpler functions, which satisfy some, but not all of the necessary boundary conditions have been determined §2*d*. These functions are relatively easy to compute, but the method will then require some integrations to be taken along the crack or notch sides. For the special case of crack tip stress analysis 'complete' dual functions can be obtained in the Laplace transform domain by the methods of [AC], [CA] (§2*b*(ii)), thus enabling the ancillary integrals to be removed from the crack tip. These ancillary solutions, although complicated and violating the physical edge conditions, can also be used, as eigensolutions when singular perturbation methods are used: a fact we hope to exploit in the future. For the real time reciprocal theorem, dual functions are given in §4*a* for the heat equation and 'incomplete' dual functions are derived in §4*a* for notch problems.

In §3 a different method is considered based on a 'pseudo' energy–momentum tensor. For the fully coupled poroelastic equations, this method enables a simple determination of a relation between near crack tip stress and pore pressure gradient intensity factors (cf. Atkinson & Smelser (1982) for applications in thermo-viscoelasticity). Moreover, by careful choice of an appropriate problem – a displacement-free strip boundary with uniform pore pressure applied on its sides and a stress-free impermeable, central crack – it is possible to let the strip width tend to infinity and recover the problem of a semi-infinite stress free crack loaded with a uniform pore pressure gradient in an infinite medium. This leads to the result (174) §3*a*(v) which serves as a quite remarkable check on the complicated analysis of [CA].

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Appendix 1. Transform results

If F denotes the Fourier transform operator, then

$$F^{-1}(1/\xi_+^{n+\frac{1}{2}}) = x^{n-\frac{1}{2}}H(x)/\mathrm{i}_+^{n+\frac{1}{2}}\gamma(n+\tfrac{1}{2}), \tag{200}$$

where n is an integer, γ is the gamma function defined to be

$$\gamma(n+1) = \int_0^\infty t^n \,\mathrm{e}^{-t}\,\mathrm{d}t \tag{201}$$

and the result that $\gamma(z)\,\gamma(1-z) = \pi \operatorname{cosec}(\pi z)$ has been used. Hence

$$F_+(x^{n-\frac{1}{2}}) = \gamma(n+\tfrac{1}{2})\,(\mathrm{i}/\xi)_+^{n+\frac{1}{2}} \tag{202}$$

and $F_-((-x)^{n-\frac{1}{2}}) = F_+^*(x^{n-\frac{1}{2}})$, i.e. the complex conjugate of the equivalent plus transform.

Defining $\Gamma^2 = \xi^2 + a^2$ where Γ has branch cuts from $\pm\mathrm{i}a$ to $\pm\mathrm{i}\infty$ we can evaluate following inverse Fourier transform

$$F^{-1}\left(\frac{\mathrm{e}^{-\Gamma y}}{\Gamma_+}\right) = \frac{1}{(2\pi\mathrm{i})_+^{\frac{1}{2}}}\frac{y\,\mathrm{e}^{-ar}}{r(r-x)^{\frac{1}{2}}} = \frac{1}{(\pi\mathrm{i})_+^{\frac{1}{2}}}\frac{\cos\frac{1}{2}\theta}{r^{\frac{1}{2}}}\mathrm{e}^{-ar}. \tag{203}$$

This is obtained by collapsing the inversion integral around the branch cuts for Γ and then evaluating the resulting definite integrals using 3.962 of Gradshteyn & Ryzhik (1980). Note that $r^2 = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

The following recurrence relations between the $K_\nu(z)$ from 8.486 of Gradshteyn & Ryzhik (1980) are also required in the text

$$\mathrm{d}K_\nu(z)/\mathrm{d}z = (\nu/z)\,K_\nu(z) - K_{\nu+1}(z), \tag{204}$$

$$\mathrm{d}K_\nu(z)/\mathrm{d}z = -K_{\nu-1}(z) - (\nu/z)\,K_\nu(z). \tag{205}$$

Appendix 2

Useful results from [AC] and [CA] are noted here for use in the text.

Define

$$N(\xi) = \xi^2 - |\xi|\,(\xi^2+1)^{\frac{1}{2}} + \overline{\eta}. \tag{206}$$

From contour integration we find

$$\ln\left(\frac{N_-(\xi)}{-N_0}\right) = -\frac{1}{\pi}\int_0^1 \arctan\left(\frac{p(1-p^2)^{\frac{1}{2}}}{p^2-\overline{\eta}}\right)\frac{\mathrm{d}p}{p+\mathrm{i}\xi}. \tag{207}$$

Although initially defined in the lower half-plane, by analytic continuation this defines a function valid in the whole complex plane except for the branch cut $\mathrm{i}\epsilon$ to i. As the above integral has finite range it can be formally expanded for large ξ to give the following asymptotic result.

$$\frac{N_-(\xi)}{-N_0} \sim 1 + \frac{\mathrm{i}}{\pi\xi}\int_0^1 \arctan\left(\frac{p(1-p^2)^{\frac{1}{2}}}{p^2-\overline{\eta}}\right)\mathrm{d}p. \tag{208}$$

As $N(\xi) \sim -N_0 + O(\xi^{-2})$ for $|\xi| \to \infty$ this implies that,

$$N_+(\xi) \sim 1 - \frac{\mathrm{i}}{\pi\xi}\int_0^1 \arctan\left(\frac{p(1-p^2)^{\frac{1}{2}}}{p^2-\overline{\eta}}\right)\mathrm{d}p. \tag{209}$$

The results above imply that $N_+(0) = ((1-\nu)/(1-\nu_u))^{\frac{1}{2}}$.

Define

$$k_+(\xi) + k_-(\xi) = 1/N_-(\xi)\,\Gamma_+(\xi)\,\xi_+^{\frac{1}{2}}. \tag{210}$$

We note here that in the sum split of $k(\xi)$ in [AC] there is a typing error in (197) which should read

$$\frac{1}{2\pi\mathrm{i}}\int_C \frac{K(z)}{z-\xi}\mathrm{d}z = \frac{1}{\pi}\int_0^1 \frac{\mathrm{d}y}{(\mathrm{i}y+\xi)\,y^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}\left(\frac{1}{N_-(-\mathrm{i}y)} - \frac{N_+(0)}{\overline{\eta}}\right). \tag{211}$$

As $|\xi| \to \infty$

$$k_+ \to c_0 + (N_+(0)/\overline{\eta} - d)\,\xi^{-1} - (\mathrm{i}N_+(0)/2\overline{\eta} + \varpi)\,\xi^{-2} + \dots \tag{212}$$

and

$$k_- \to -c_0 + \left(d - \frac{1}{N_0} - \frac{N_+(0)}{\bar{\eta}}\right)\xi^{-1} + \left(\varpi + \tfrac{1}{2}\mathrm{i}\left(\frac{N_+(0)}{\bar{\eta}} + \frac{1}{N_0} + \frac{I}{N_0}\right)\right)\xi^{-2} + \ldots, \tag{213}$$

where

$$I = \frac{2}{\pi}\int_0^1 \arctan\left(\frac{p(1-p^2)^{\frac{1}{2}}}{p^2-\bar{\eta}}\right)\mathrm{d}p, \tag{214}$$

$$c_0 = \frac{-1}{\mathrm{i}\pi}\int_0^1 \frac{\mathrm{d}y}{y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}}\left(\frac{1}{N_-(-\mathrm{i}y)} - \frac{N_+(0)}{\bar{\eta}}\right), \tag{215}$$

$$d = \frac{-1}{\pi}\int_0^1 \frac{\mathrm{d}y}{y^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}\left(\frac{1}{N_-(-\mathrm{i}y)} - \frac{N_+(0)}{\bar{\eta}}\right), \tag{216}$$

$$\varpi = \frac{\mathrm{i}}{\pi}\int_0^1 \frac{y^{\frac{1}{2}}\mathrm{d}y}{(1-y)^{\frac{1}{2}}}\left(\frac{1}{N_-(-\mathrm{i}y)} - \frac{N_+(0)}{\bar{\eta}}\right). \tag{217}$$

Define

$$\bar{N}(\xi) = (\xi^2/\Gamma)(\Gamma - |\xi|) - \bar{\eta}. \tag{218}$$

The Cauchy representations for $\bar{N}_\pm$ can be deduced as

$$\pm\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty} \frac{\ln(\bar{N}(z)/N_0)}{z-\xi}\,\mathrm{d}z = \begin{cases}\ln \bar{N}_+(\xi), \\ \ln(\bar{N}_-(\xi)/N_0),\end{cases} \tag{219}$$

noting that $\bar{N}(\xi)$ has zeros in the cut plane at $\xi = \pm\mathrm{i}\alpha$ where α is given by

$$\alpha = \left(\bar{\eta}\frac{(2-\bar{\eta}) + (\bar{\eta}^2 + 4\bar{\eta})^{\frac{1}{2}}}{2(2\bar{\eta}-1)}\right)^{\frac{1}{2}}, \tag{220}$$

we can deduce that $\bar{N}(\xi)$ has branch cuts from $\pm\mathrm{i}\epsilon$ to $\pm\mathrm{i}\alpha$. The branch cut from 0 to $\pm\mathrm{i}$ is due to the cuts for the functions contained in $\bar{N}(\xi)$. As $\bar{N}(\xi)$ is negative from $\pm\mathrm{i}$ to $\pm\mathrm{i}\alpha$ the logarithm is not defined; however, from analytic continuation the logarithm is defined correctly outside this interval, so the branch cut extends to $\pm\mathrm{i}\alpha$. This can be checked numerically by following the argument of the complex logarithm in the complex plane.

$$\ln\left(\frac{\bar{N}_-(\xi)}{N_0}\right) = \frac{1}{\pi}\int_0^1 \arctan\left(\frac{p^3}{(p^2+\bar{\eta})(1-p^2)^{\frac{1}{2}}}\right)\frac{\mathrm{d}p}{p+\mathrm{i}\xi} + \ln\left(\frac{\alpha+\mathrm{i}\xi}{1+\mathrm{i}\xi}\right). \tag{221}$$

Although initially defined in the lower half-plane, by analytic continuation this defines a function valid in the whole complex plane except for the branch cut from $\mathrm{i}\epsilon$ to $\mathrm{i}\alpha$.

The result for $\bar{N}_+$ is the complex conjugate of $N_0\bar{N}_-(\xi)$. The expression for $\bar{N}_-(\xi)$ can be evaluated in the limit as $|\xi| \to \infty$

$$\frac{\bar{N}_-(\xi)}{N_0} \sim 1 + \frac{1}{\mathrm{i}\xi}\left(\alpha - 1 + \frac{1}{\pi}\int_0^1 \arctan\left(\frac{p^3}{(p^2+\bar{\eta})(1-p^2)^{\frac{1}{2}}}\right)\mathrm{d}p\right) + \ldots. \tag{222}$$

Note that $\bar{N}_+(0) = N_+(0) = ((1-\nu)/(1-\nu_u))^{\frac{1}{2}}$. This is checked numerically in both [CA], [AC] and in [CA] is used in the asymptotic limit as $t \to 0$ to prove that in the unmixed cases the stress intensity factors are the same as those for a crack tip embedded in a drained inclusion.

Define

$$\bar{k}(\xi) = \Gamma_+/\bar{N}_-\,\xi_+^{\frac{3}{2}} = \bar{k}_+(\xi)+\bar{k}_-(\xi), \tag{223}$$

where k has branch cuts from $\pm\mathrm{i}\epsilon$ to $\pm\mathrm{i}$ and a pole at $\mathrm{i}\alpha$.

$$\bar{K}(\xi) = \bar{k}(\xi)-\frac{\Gamma_+}{\xi_+^{\frac{3}{2}}\bar{N}_-(0)} = \frac{\Gamma_+}{\xi_+^{\frac{3}{2}}}\left(\frac{1}{\bar{N}_-(\xi)}-\frac{1}{\bar{N}_-(0)}\right), \tag{224}$$

$$\bar{K}_+(\xi) = \frac{\mathrm{i}}{\pi}\int_0^1 \frac{(1-y)^{\frac{1}{2}}\,\mathrm{d}y}{(y-\mathrm{i}\xi)\,y^{\frac{3}{2}}}\left(\frac{1}{\bar{N}_-(-\mathrm{i}y)}-\frac{1}{\bar{N}_-(0)}\right), \tag{225}$$

$$\bar{K}_-(\xi) = \frac{\Gamma_+}{\xi_+^{\frac{3}{2}}}\left(\frac{1}{\bar{N}_-}-\frac{1}{\bar{N}_-(0)}\right)-K_+(\xi). \tag{226}$$

The asymptotic behaviours of $\bar{k}_+, \bar{k}_-$ as $|\xi|\rightarrow\infty$ are

$$\bar{k}_+(\xi) \sim \frac{1}{\xi}\left(\frac{1}{\bar{N}_-(0)}+\bar{d}\right)+\frac{\mathrm{i}}{\xi^2}\left(\frac{1}{2\bar{N}_-(0)}+\bar{\chi}\right)+\dots, \tag{227}$$

$$\bar{k}_- \sim \frac{1}{\xi}\left(\frac{1}{N_0}-\frac{1}{\bar{N}_-(0)}-\bar{d}\right)+\frac{\mathrm{i}}{\xi^2}\left(\frac{1}{2}\left(\frac{1}{N_0}-\frac{1}{\bar{N}_-(0)}\right)+\frac{\varpi}{N_0}-\bar{\chi}\right)+\dots, \tag{228}$$

where

$$\bar{d} = \frac{-1}{\pi}\int_0^1 \frac{(1-y)^{\frac{1}{2}}}{y^{\frac{3}{2}}}\left(\frac{1}{\bar{N}_-(-\mathrm{i}y)}-\frac{1}{\bar{N}_-(0)}\right)\mathrm{d}y, \tag{229}$$

$$\bar{\varpi} = \alpha-1+\frac{1}{\pi}\int_0^1 \arctan\left(\frac{p^3}{(p^2+\bar{\eta})\,(1-p^2)^{\frac{1}{2}}}\right)\mathrm{d}p, \tag{230}$$

$$\bar{\chi} = \frac{1}{\pi}\int_0^1 \frac{(1-y)^{\frac{1}{2}}}{y^{\frac{1}{2}}}\left(\frac{1}{\bar{N}_-(-\mathrm{i}y)}-\frac{1}{\bar{N}_-(0)}\right)\mathrm{d}y. \tag{231}$$

Appendix 3

The displacements, stresses and pore pressure for tension are

$$u_1 = \frac{\partial\Psi}{\partial x}+\frac{y}{(1-2\nu_u)}\frac{\partial^2\Psi}{\partial x\,\partial y}+\frac{G_u}{G}\frac{\partial\Phi}{\partial x}, \tag{232}$$

$$u_2 = \frac{-2(1-\nu_u)}{(1-2\nu_u)}\frac{\partial\Psi}{\partial y}+\frac{y}{(1-2\nu_u)}\frac{\partial^2\Psi}{\partial y^2}+\frac{G_u}{G}\frac{\partial\Phi}{\partial y}, \tag{233}$$

$$p = 2Q\alpha\,\partial^2\Psi/\partial y^2+(c/\kappa)\,\nabla^2\Phi, \tag{234}$$

$$\sigma_{11} = 2G\left(\frac{\partial^2\Psi}{\partial x^2}-\frac{2\nu_u}{(1-2\nu_u)}\frac{\partial^2\Psi}{\partial y^2}+\frac{y}{(1-2\nu_u)}\frac{\partial^3\Psi}{\partial x^2\,\partial y}-\frac{G_u}{G}\frac{\partial^2\Phi}{\partial y^2}\right), \tag{235}$$

$$\sigma_{12} = 2G\left(\frac{y}{(1-2\nu_u)}\frac{\partial^3\Psi}{\partial x\,\partial y^2}+\frac{G_u}{G}\frac{\partial^2\Phi}{\partial x\,\partial y}\right), \tag{236}$$

$$\sigma_{22} = 2G\left(\frac{-1}{(1-2\nu_u)}\frac{\partial^2\Psi}{\partial y^2}+\frac{y}{(1-2\nu_u)}\frac{\partial^3\Psi}{\partial y^3}-\frac{G_u}{G}\frac{\partial^2\Phi}{\partial x^2}\right). \tag{237}$$

In the Fourier transform domain these are

$$u_1 = \left(-\mathrm{i}\xi B_1 + \frac{\mathrm{i}\xi y\,|\xi|\,B_1}{(1-2\nu_u)} - \frac{\mathrm{i}\xi G_u}{G} A_1\right)\mathrm{e}^{-|\xi|y} - \frac{\mathrm{i}\xi G_u}{G} A_2\,\mathrm{e}^{-\Gamma y}, \tag{238}$$

$$u_2 = \mathrm{e}^{-|\xi|y}(B_1\,(1-2\nu_u)^{-1}\,(2(1-\nu_u)\,|\xi| + y\xi^2) - A_1(G_u/G)\,|\xi|) - (G_u/G)\,A_2\,\Gamma\mathrm{e}^{-\Gamma y}, \tag{239}$$

$$p = 2Q\alpha\xi^2 B_1\,\mathrm{e}^{-|\xi|y} + (c/\kappa)\,A_2(\Gamma^2-\xi^2)\,\mathrm{e}^{-\Gamma y}, \tag{240}$$

$$\sigma_{11} = 2G(1-2\nu_u)^{-1}\,\xi^2 B_1\,\mathrm{e}^{-|\xi|y}(y\,|\xi|-1) - 2G_u(A_1\,\xi^2\,\mathrm{e}^{-|\xi|y} + A_2\,\Gamma^2\,\mathrm{e}^{-\Gamma y}), \tag{241}$$

$$\sigma_{12} = -2G\,y\mathrm{i}\xi^3\,B_1\,\mathrm{e}^{-|\xi|y}(1-2\nu_u)^{-1} + 2G_u\,\mathrm{i}\xi(|\xi|A_1\,\mathrm{e}^{-|\xi|y} + \Gamma A_2\,\mathrm{e}^{-\Gamma y}), \tag{242}$$

$$\sigma_{22} = -2G(1-2\nu_u)^{-1}\,\xi^2 B_1\,\mathrm{e}^{-|\xi|y}(1+y|\xi|) + 2G_u\,\xi^2(A_1\,\mathrm{e}^{-|\xi|y} + A_2\,\mathrm{e}^{-\Gamma y}). \tag{243}$$

The displacements, stresses and pore pressure for shear are

$$u_1 = \frac{\partial\Psi}{\partial x} + \frac{y}{2(1-\nu_u)}\frac{\partial^2\Psi}{\partial x\,\partial y} + \frac{G_u}{G}\frac{\partial\Phi}{\partial x}, \tag{244}$$

$$u_2 = \frac{-(1-2\nu_u)}{2(1-\nu_u)}\frac{\partial\Phi}{\partial y} + \frac{y}{2(1-\nu_u)}\frac{\partial^2\Psi}{\partial y^2} + \frac{G_u}{G}\frac{\partial\Phi}{\partial y}, \tag{245}$$

$$\sigma_{11} = 2G\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{y}{2(1-2\nu_u)}\frac{\partial^3\Psi}{\partial x^2\,\partial y} - \frac{\nu_u}{(1-\nu_u)}\frac{\partial^2\Psi}{\partial y^2}\right) - 2G_u\frac{\partial^2\Phi}{\partial y^2}, \tag{246}$$

$$\sigma_{12} = 2G\left(\frac{1}{2(1-\nu_u)}\frac{\partial^2\Psi}{\partial x\,\partial y} + \frac{y}{2(1-\nu_u)}\frac{\partial^3\Psi}{\partial x\,\partial^2 y}\right) + 2G_u\frac{\partial^2\Phi}{\partial x\,\partial y}, \tag{247}$$

$$\sigma_{22} = 2G\left(\frac{y}{2(1-\nu_u)}\frac{\partial^3\Psi}{\partial y^3}\right) - 2G_u\frac{\partial^2\Phi}{\partial x^2}, \tag{248}$$

$$p = \frac{c}{\kappa}\nabla^2\Phi + \frac{\alpha Q(1-2\nu_u)}{(1-\nu_u)}\frac{\partial^2\Psi}{\partial y^2}. \tag{249}$$

In Fourier transform space:

$$u_1 = \mathrm{e}^{-|\xi|y}\left(B_1\left(\frac{\mathrm{i}\xi|\xi|y}{2(1-\nu_u)} - \mathrm{i}\xi\right) - A_1\frac{\mathrm{i}\xi G_u}{G}\right) - A_2\frac{\mathrm{i}\xi G_u}{G}\,\mathrm{e}^{-\Gamma y}, \tag{250}$$

$$u_2 = \frac{\mathrm{e}^{-|\xi|y}B_1}{2(1-\nu_u)}((1-2\nu_u)\,|\xi| + y\xi^2) - \frac{G_u}{G}(A_1\,|\xi|\,\mathrm{e}^{-|\xi|y} + A_2\,\Gamma\mathrm{e}^{-\Gamma y}) \tag{251}$$

$$\sigma_{11} = 2G\left(\frac{-\xi^2 B_1\,\mathrm{e}^{-|\xi|y}}{(1-\nu_u)} + \frac{y\xi^2\,|\xi|\,B_1\,\mathrm{e}^{-|\xi|y}}{2(1-\nu_u)}\right) - 2G_u(A_1\,\xi^2\,\mathrm{e}^{-|\xi|y} + A_2\,\Gamma^2\,\mathrm{e}^{-\Gamma y}), \tag{252}$$

$$\sigma_{12} = \frac{2G\mathrm{i}\xi B_1\,\mathrm{e}^{-|\xi|y}}{2(1-\nu_u)}(|\xi| - y\xi^2) + 2G_u\,\mathrm{i}\xi(A_1\,|\xi|\,\mathrm{e}^{-|\xi|y} + A_2\,\Gamma\mathrm{e}^{-\Gamma y}), \tag{253}$$

$$\sigma_{22} = 2G\left(\frac{-y\xi^2\,|\xi|\,B_1\,\mathrm{e}^{-|\xi|y}}{2(1-\nu_u)}\right) + 2G_u\,\xi^2(A_1\,\mathrm{e}^{-|\xi|y} + A_2\,\mathrm{e}^{-\Gamma y}), \tag{254}$$

$$p = \frac{c}{\kappa}(\Gamma^2-\xi^2)\,A_2\,\mathrm{e}^{-\Gamma y} + \frac{\alpha Q(1-2\nu_u)\,B_1\,\xi^2}{(1-\nu_u)}\,\mathrm{e}^{-|\xi|y}. \tag{255}$$

Appendix 4. The near fields (elastic eigensolutions)

The following results are well known but are given here as the results are used widely in the text.

To get the tensile stress intensity factors in the neighbourhood of the crack tip it is necessary to know the structure of the stress, pressure and displacement fields there. In the near vicinity of the crack tip, i.e. as $r \to 0$, the term in $\nabla^2 \bar{p}$ in (21) is dominant. Thus the equations become mathematically equivalent to those of uncoupled thermal stress. In the neighbourhood of the stress singularity the pressure equation can be assumed to reduce to

$$\nabla^2 \bar{p} = 0. \tag{256}$$

We also introduce a stress function ϕ such that

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right). \tag{257}$$

So the elastic equilibrium equations reduce to

$$\nabla^4 \phi = 0. \tag{258}$$

In what follows all the field variables are assumed to be in the Laplace transform domain and the notation $\bar{a}$ will be used to denote the Mellin transform of a.

Using the Mellin transform define

$$\bar{\phi} = \int_0^{+\infty} r^{\lambda-2}\, \phi(r,\theta,s)\,\mathrm{d}r, \quad \bar{\sigma}_{ij} = \int_0^{\infty} r^{\lambda}\, \sigma_{ij}(r,\theta,s)\,\mathrm{d}r. \tag{259}$$

Then

$$\bar{\sigma}_{rr} = -(\lambda-1)\,\bar{\phi} + \mathrm{d}^2\bar{\phi}/\mathrm{d}\theta^2, \quad \bar{\sigma}_{r\theta} = \lambda \mathrm{d}\bar{\phi}/\mathrm{d}\theta, \quad \bar{\sigma}_{\theta\theta} = \lambda(\lambda-1)\,\bar{\phi}. \tag{260}$$

To invert

$$\phi = \frac{1}{2\pi \mathrm{i}}\int_{Br} \bar{\phi} r^{-(\lambda-1)}\,\mathrm{d}\lambda, \quad \sigma_{ij} = \frac{1}{2\pi \mathrm{i}}\int_{Br} \bar{\sigma}_{ij}\, r^{-\lambda-1}\,\mathrm{d}\lambda, \tag{261}$$

here Br is the Bromwich inversion contour. To deduce the angular form of the stresses in the neighbourhood of the crack tip we take the following boundary conditions on $\theta = \pm\beta$:

$$\sigma_{\theta\theta}(r, \pm\beta, s) = \begin{cases} \bar{\sigma}(s) & \text{for} \quad r < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{262}$$

$$\sigma_{r\theta}(r, \pm\beta, s) = 0\,\forall r. \tag{263}$$

We define the Mellin transform of $\bar{\sigma}$ as

$$\bar{\sigma} = \sigma(\lambda+1)^{-1} \quad \text{for} \quad \Re(\lambda) > -1. \tag{264}$$

As the tensile problem is symmetric the solution of the Mellin transformed biharmonic equation is

$$\bar{\phi} = A \cos(\lambda+1)\,\theta + C\cos(\lambda-1)\,\theta. \tag{265}$$

From the boundary conditions

$$A = \frac{-\bar{\sigma}\sin((\lambda-1)\beta)}{\lambda(\sin 2\lambda\beta+\lambda\sin(2\beta))}, \quad C = \frac{\bar{\sigma}(\lambda+1)\sin((\lambda+1)\beta)}{\lambda(\sin 2\lambda\beta+\lambda\sin(2\beta))(\lambda-1)}. \tag{266}$$

The displacements can be deduced from

$$\left.\begin{aligned} 2G\bar{u}_r &= (\lambda-1)\bar{\phi}+(1-\nu)\,\mathrm{d}\bar{\psi}/\mathrm{d}\theta, \\ 2G\bar{u}_\theta &= -\mathrm{d}\bar{\phi}/\mathrm{d}\theta-(1-\nu)(\lambda+1)\bar{\psi}, \end{aligned}\right\} \tag{267}$$

where $$\bar{\psi} = 4A(\lambda+1)^{-1}\sin(\lambda+1)\theta \tag{268}$$

with $$u_i = \frac{1}{2\pi \mathrm{i}}\int_{Br}\bar{u}_i r^{-\lambda}\,\mathrm{d}\lambda. \tag{269}$$

The transforms can be simply inverted using the residue theorem by noting the simple poles at the zeros of

$$\sin 2\lambda\beta+\lambda\sin(2\beta), \tag{270}$$

which for the crack ($\beta = \pi$) are $\lambda = \pm\frac{1}{2}(2n+1), n = 0, \pm 1, \ldots$ (and at $\lambda = -1$ for $\sigma_{\theta\theta}$). Note that we are here only considering angles β such that $\beta^* < \beta \leqslant \pi$ and β^* is the root of $\tan 2\beta = -2\beta$. The angular behaviour of the field variables can then be quickly deduced. In general

$$p = K_2(s)(2\pi)^{-\frac{1}{2}} r^{(2n+1)\pi/2\beta}\cos((2n+1)\pi\theta/2\beta), \tag{271}$$

$$\sigma_{rr} = (1/2(2\pi)^{\frac{1}{2}})K_1(s)((\lambda+3)\sin(\lambda-1)\beta\cos(\lambda+1)\theta - (\lambda+1)\sin(\lambda+1)\beta\cos(\lambda-1)\theta)\,r^{-\lambda-1}, \tag{272}$$

$$\sigma_{r\theta} = (1/2(2\pi)^{\frac{1}{2}})K_1(s)(\lambda+1)(\sin(\lambda+1)\theta\sin(\lambda-1)\beta-\sin(\lambda+1)\beta\sin(\lambda-1)\theta)\,r^{-\lambda-1}, \tag{273}$$

$$\sigma_{\theta\theta} = (1/2(2\pi)^{\frac{1}{2}})K_1(s)((\lambda+1)\cos(\lambda-1)\theta\sin(\lambda+1)\beta - (\lambda-1)\sin(\lambda-1)\beta\cos(\lambda+1)\theta)\,r^{-\lambda-1}, \tag{274}$$

$$2Gu_r = (K_1(s)/2\lambda(2\pi)^{\frac{1}{2}})(-(\lambda-1+4(1-\nu))\sin(\lambda-1)\beta\cos(\lambda+1)\theta + (\lambda+1)\sin(\lambda+1)\beta\cos(\lambda-1)\theta)\,r^{-\lambda}, \tag{275}$$

$$2Gu_\theta = (K_1(s)/2\lambda(2\pi)^{\frac{1}{2}})((\lambda+1)\sin(\lambda-1)\theta\sin(\lambda+1)\beta - (\lambda+1-4(1-\nu))\sin(\lambda-1)\beta\sin(\lambda+1)\theta)\,r^{-\lambda}. \tag{276}$$

Appendix 5

Unfortunately in Atkinson & Craster (1991) the following typing errors occurred and we take the opportunity of correcting these errors here.

$$N_0 = \tfrac{1}{2}(1-\bar{\eta}), \tag{10}$$

$$P_+ \sim \frac{c}{2G_u\kappa}\Sigma\left(\frac{\mathrm{i}N_+(0)}{2\bar{\eta}}+\varpi\right)+\frac{c^{\frac{3}{2}}\tau_0\Gamma_+(\mathrm{i}/a_1)}{s^{\frac{3}{2}}2G_u\kappa\mathrm{i}N_+(\mathrm{i}/a_1)}\left(\left(d-\frac{N_+(0)}{\bar{\eta}}\right)-\left(\frac{\mathrm{i}}{a_1}\right)\left(c_0-k_+\left(\frac{\mathrm{i}}{a_1}\right)\right)\right)\xi_+^{-\frac{3}{2}} \tag{80}$$

$$\bar{K}_2(s) = \left\{ \frac{c^{\frac{3}{2}} \tau_0 \Gamma_+(\mathrm{i}/a_1)}{s^{\frac{3}{2}} \kappa \mathrm{i} N_+(\mathrm{i}/a_1)} \left(-d + \frac{N_+(0)}{\bar{\eta}} - \left(\frac{\mathrm{i}}{a_1}\right)\left(c_0 - k_+\left(\frac{\mathrm{i}}{a_1}\right)\right)\right) + \frac{c}{2G_u \kappa} \Sigma \left(\frac{\mathrm{i}N_+(0)}{2\bar{\eta}} + \varpi \right) \right\} \mathrm{e}^{\mathrm{i}\pi/4} \left(\frac{s}{c}\right)^{\frac{3}{4}} (-2\surd 2), \quad (86)$$

$$p \sim ((-\alpha V(1-2\nu) K_1 / 4\kappa G (2\pi)^{\frac{1}{2}}) \cos(3\theta/2) + K_2 \cos(\tfrac{1}{2}\theta))\, r^{\frac{1}{2}}, \quad (168)$$

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{K(z)}{z-\xi} \mathrm{d}z = \frac{1}{\pi} \int_0^1 \frac{\mathrm{d}y}{(\mathrm{i}y+\xi)\, y^{\frac{1}{2}} (1-y)^{\frac{1}{2}}} \left(\frac{1}{N_-(-\mathrm{i}y)} - \frac{N_+(0)}{\bar{\eta}} \right), \quad (197)$$

$$\bar{\eta} = (1-\nu) \frac{(1+(1-2\nu)\,\beta^2\, T_0/Gc_\epsilon)}{(1-2\nu)^2 \beta^2\, T_0/Gc_\epsilon}. \quad (219)$$

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